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Sur la Croissance des Séries Entières

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Étant donné une série entière

$$f(z)=a_0+a_1z+a_2z^2+\cdots\cdots$$

convergente dans un cercle $|z|<R\,(0<R\leqq\infty)$, nous connaissons d'après MM. Poincaré, Hadamard, Borel, Lindelöf, Valiron etc., diverses relations entre la croissance pour $r\to R$ du module maximum

$$M(r)=\max_{|z|=r}|f(z)|$$

et celle de la suite

$$|a_0|,\ |a_1|,\cdots\cdots,\ |a_n|,\cdots\cdots .^{1}$$

Nous allons chercher à les obtenir toutes d'une même manière.

À cet effet, introduisons le terme maximum

$$m(r)=\max_{n\geqq 0}|a_n|r^n,$$

et nous avons les inégalités simples

$$(1)\qquad m(r)\leqq M(r)\leqq m(r')\frac{1}{1-\frac{r}{r'}}\quad[0\leqq r<r'<R].^{2}$$

D'autre part, M. Valiron s'est mis à utiliser le rang de terme maximum $N(r)$ c'est-à-dire le plus grand entier n tel que $|a_n|r^n=m(r)$[3]; on a

$$\log m(e^x)=\int^{x}N(e^x)dx\quad[-\infty<x<\log R],$$

où $N(e^x)$ est une fonction non décroissante, continue à droite, prenant les seules valeurs des entiers positifs ou nul, mais quelconque d'ailleurs[4].

Dans le présent mémoire, nous donnerons d'abord (§1) des rela-

1. À propos des résultats classiques pour les fonctions entières, *voir* G. Valiron, Fonctions entières et méromorphes d'une variable, p. 7-8. Des mémoires plus récents : Okamura, Tôhoku Math. J. **38** (1933), p. 129 ; King-Lai Hiong, J. Math. Pures Appl. 9 série **15** (1935), p. 233. Par rapport aux séries entières à rayon de convergence fini, *voir* Valiron, Bull. Soc. Math. France **44** (1916), p. 45 ; Beuermann, Math. Z. **33** (1931), p. 98 ; Fujiwara, Proc. Imp. Acad. Jap. **8** (1932), p. 220 ; King-Lai Hiong, *loc. cit.*

2. Qui ont été signalées par M. Borel, Acta Math. **20** (1897), p. 383, tandis que M. Hadamard a donné une énonciation géométrique précisément équivalente [Bull. Soc. Math. France **24** (1896), p. 186].

3. Dans le présent mémoire, *rejetons le cas* $a_n=0$ $(n=0,\ 1,\ 2,\ldots)$, parce qu'il n'y a rien à considérer.

4. *Voir* Valiron, Ouvrage cité, p. 6.

tions entre la croissance de $m(r)$ pour $r \to R$ et celle de la suite $\{|a_n|\}$, sous une forme explicite et aussi générale que possible. Pour cela, nous introduisons une fonction $g(x)$ *continue dans un intervalle* $X_0 \leqq x < X = \log R$, *dont la dérivée* $g'(x)$ *est aussi continue, croissante au sens strict et telle que*

$$\lim_{x \to X} g'(x) = +\infty.$$

Alors, si $\lim_{r \to R} N(r) = +\infty$, on peut choisir $g(x)$ convenablement de manière que

$$\lim_{x \to X} \frac{m(e^x)}{e^{g(x)}} = 1 ;$$

c'est la raison par laquelle nous prenons une telle fonction $g(x)$, mais dans les lignes suivantes nous laisserons $g(x)$ indépendante de la croissance de $m(r)$, ne supposant pas toujours remplie cette condition. Moyennant cette auxiliaire, nous établirons, par exemple, les conditions nécessaires et suffisantes auquelles doivent satisfaire les nombres $|a_n|$ pour que l'on ait $\lim_{x \to X} \frac{m(e^x)}{e^{g(x)}} = 1$.

Nous allons montrer ensuite (§2) que l'on peut obtenir diverses relations aujourd'hui connues entre la croissance de $M(r)$ et celle de la suite $\{|a_n|\}$, partant des résultats de §1, à l'aide des inégalités (1)[1] ; nous expliquerons cela au moyen de deux exemples qui suffiront à révéler la méthode générale et à éclaircir sa portée.

§1.—Relations entre la croissance de $m(r)$ et celle de la suite $\{|a_n|\}$.

Soit $g(x)$ définie comme ci-dessus, c'est-à-dire, une fonction continue pour $X_0 \leqq x < X$, dont la dérivée $g'(x)$ est continue et croissante vers $+\infty$ pour $x \to X$; et *soit* $x = h(y)$ *la fonction inverse de* $y = g'(x)$. On a alors, d'après Stieltjes[2],

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1. Au lieu de la seconde inégalité de (1), on peut utiliser une autre :

$$M(r) < m(r)\left[N(r') + \frac{1}{1 - \frac{r}{r'}}\right] \quad [0 < r < r' < R]$$

donnée sous une forme légèrement moins précise par M. Valiron, Mémoire cité, p. 49. Démonstration de cette inégalité.—

$$M(r) \leqq \sum_{n=0}^{\infty} |a_n| r^n = \sum_{n=0}^{N(r')-1} |a_n| r^n + \sum_{n=N(r')}^{\infty} |a_n| r^n \leqq N(r') m(r) + \sum_{n=N(r')}^{\infty} |a_n| r'^n \left(\frac{r}{r'}\right)^n,$$

où le second terme $< \sum_{n=N(r')}^{\infty} |a_{N(r')}| r'^{N(r')} \left(\frac{r}{r'}\right)^n = |a_{N(r')}| r'^{N(r')} \sum_{n=0}^{\infty} \left(\frac{r}{r'}\right)^n \leqq m(r) \frac{1}{1 - \frac{r}{r'}}$. (C. q. f. d.) Mais nous préférons l'inégalité (1) car elle est plus simple et directe.

2. Ann. Fac. Sci. Univ. Toulouse **8** (1894), p. J. 71-72 ; *voir* Hobson, The theory of functions of a real variable, vol. 1, 3rd. ed., p. 538-539.

$$xg'(x)-g(x)=xg'(x)-\int g'(x)dx=\int xdg'(x) \quad [X_0\leqq x<X],$$

ou, en le transformant selon $x=h(y)$ et en désignant par $k(y)$ le premier membre, on a

$$(2) \qquad k(y)=yh(y)-g[h(y)]=\int h(y)dy \quad [g'(X_0)\leqq y<+\infty].$$

Grâce à cette formule, nous obtenons l'identité :

$$(3) \qquad \frac{|a_n|e^{nx}}{e^{g(x)}}=|a_n|e^{k(n)-\int_y^n[h(t)-h(y)]dt}$$

$$[y=g'(x),\ X_0\leqq x<X,\ n\geqq g'(X_0)],^1$$

qui est, avec (2), fondamentale en nos raisonnements pour établir les propositions suivantes :

I. *Si* $\dfrac{m(e^x)}{e^{g(x)}}<A$ $[X_0\leqq x<X]$, *on a alors* $|a_n|e^{k(n)}<A \quad [n\geqq g'(X_0)]$.

Démonstration.—D'après (3), on a pour $x=h(n)\geqq X_0$,

$$\frac{|a_n|e^{nx}}{e^{g(x)}}=|a_n|e^{k(n)},$$

dont le premier membre est, par hypothèse, inférieur à A. (C. q. f. d.)

II. *Si* $|a_n|e^{k(n)}<A$ $[n\geqq n_0\geqq g'(X_0)]$,[2] *on a alors* $\dfrac{m(e^x)}{e^{g(x)}}<A$ *pour* x *tel que* $N(e^x)\geqq n_0$.

Dém.—On a, par (3),

$$\frac{|a_n|e^{nx}}{e^{g(x)}}\leqq|a_n|e^{k(n)} \quad [n\geqq g'(X_0)],$$

donc

$$\frac{m(e^x)}{e^{g(x)}}\leqq|a_{N(e^x)}|e^{k[N(e^x)]},$$

où le second membre est, par hypothèse, inférieur à A. (C. q. f. d.)

III. *Supposant que* $\lim\limits_{r\to R}N(r)=+\infty$, *on a*

$$\overline{\lim_{x\to X}}\frac{m(e^x)}{e^{g(x)}}=\overline{\lim_{n\to\infty}}|a_n|e^{k(n)}.$$

C'est une conséquence de I et de II, en faisant $X_0\to X$ et $n_0\to\infty$.

IV. *Soient* $\bar g(x)$, $\bar g'(x)$, $\bar h(y)$, $\bar k(y)$ *de la même nature que* $g(x)$, $g'(x)$, $h(y)$, $k(y)$; *si*

$$\lim_{r\to R}N(r)=+\infty,\ e^{\bar g(x)}>m(e^x)>e^{g(x)} \quad [X_0\leqq x<X],$$

il y a alors une suite croissante des entiers $n_1, n_2,\ldots,n_p,\ldots$, *tels que*

1. J'ai autrefois déduit ces formules (2) et (3) sans utiliser l'intégrale de Stieltjes [ces Memoirs, A **15** (1932), p. 240-242].

2. De cette hypothèse il résulte que la série $f(z)$ est convergente pour $|z|<e^X$.

$$|a_{n_p}|e^{k(n_p)}>1,\quad \frac{\int_{\frac{1}{n_{p+1}}}^{\frac{1}{n_p}} \bar{g}\left[\bar{h}\left(\frac{1}{t}\right)\right]dt}{\frac{1}{n_p}-\frac{1}{n_{p+1}}}>g\left(\frac{\int_{n_p}^{n_{p+1}}\bar{h}(t)dt}{n_{p+1}-n_p}\right).$$

Dém.—En vertu de II, le maximum de $|a_n|e^{nx}$, pour n tel que $|a_n|e^{k(n)}\leqq 1$ $[n\geqq g'(X_0)]$, n'est pas supérieur à $e^{g(x)}$ pour x tel que $N(e^x)\geqq g'(X_0)$. Donc, en désignant par $n_1, n_2,\ldots,n_p,\ldots$, les indices n tels que $|a_n|e^{k(n)}>1$ $[n\geqq g'(X_0)]$, nous avons

$$\max_{p\geq 1}|a_{n_p}|e^{n_p x}=m(x)>e^{g(x)}$$

pour x assez voisin de X. Par conséquent, on a, d'après (3),

$$|a_{n_p}|e^{k(n_p)-\int_y^{n_p}[h(t)-h(y)]dt}>1 \quad [y=g'(x)] \text{ avec } p\to\infty\ (x\to X).$$

D'autre part, I entraîne, par hypothèse, $|a_n|e^{\bar{k}(n)}<1$ $[n\geqq\bar{g}'(X_0)]$, et il arrive, pour y suffisamment grand,

$$e^{-\bar{k}(n_p)+k(n_p)-\int_y^{n_p}[h(t)-h(y)]dt}>1 \text{ avec } p\to\infty\ (y\to\infty),$$

ou, en posant

$$\Theta(u,\ y)=-\bar{k}(u)+k(u)-\int_y^u[h(t)-h(y)]dt,$$

on a

(4) $$\Theta(n_p,\ y)>0 \text{ avec } p\to\infty\ (y\to\infty).$$

Or, nous avons d'après (2),

(5) $$\Theta(u,\ y)=-\bar{k}(u)+uh(y)-g[h(y)],$$

d'où $$\Theta'_u(u,\ y)=-\bar{h}(u)+h(y);$$

donc, $\Theta(u,\ y)$ est, comme fonction d'une variable u, croissante pour $\bar{g}'(X_0)\leqq u\leqq\bar{g}'[h(y)]$ et décroissante pour $\bar{g}'[h(y)]\leqq u<+\infty$; elle atteint le maximum pour $u=\bar{g}'[h(y)]$. Par conséquent, si nous prenons un nombre y_p tel que $\Theta(n_p,\ y_p)=\Theta(n_{p+1},\ y_p)$, c'est-à-dire que

$$h(y_p)=\frac{\bar{k}(n_{p+1})-\bar{k}(n_p)}{n_{p+1}-n_p}$$

ou que

$$y_p=g'\left(\frac{\int_{n_p}^{n_{p+1}}\bar{h}(t)dt}{n_{p+1}-n_p}\right),$$

alors $$\ldots<\Theta(n_{p-1},\ y_p)<\Theta(n_p,\ y_p)=\Theta(n_{p+1},\ y_p)>\Theta(n_{p+2},\ y_p)>\ldots;$$

et y_p tend vers $+\infty$. Donc, il résulte de (4) que, pour p assez grand,

(6) $$\Theta(n_p,\ y_p)>0.$$

Or, en substituant $y=y_p$ dans (5), on a

$$\Theta(n_p,\ y_p)=\frac{-\frac{\bar{k}(n_p)}{n_p}+\frac{\bar{k}(n_{p+1})}{n_{p+1}}}{\frac{1}{n_p}-\frac{1}{n_{p+1}}}-g\left(\frac{\int_{n_p}^{n_{p+1}}\bar{h}(t)dt}{n_{p+1}-n_p}\right);$$

tandis qu'en vertu de (2),

$$\frac{d}{dt}\left[-t\bar{k}\left(\frac{1}{t}\right)\right]=-\bar{k}\left(\frac{1}{t}\right)+\frac{1}{t}\bar{h}\left(\frac{1}{t}\right)=\bar{g}\left[\bar{h}\left(\frac{1}{t}\right)\right],$$

donc

$$\Theta(n_p,\ y_p)=\frac{\displaystyle\int_{\frac{1}{n_{p+1}}}^{\frac{1}{n_p}}\bar{g}\left[\bar{h}\left(\frac{1}{t}\right)\right]dt}{\dfrac{1}{n_p}-\dfrac{1}{n_{p+1}}}-g\left(\frac{\displaystyle\int_{n_p}^{n_{p+1}}\bar{h}(t)dt}{n_{p+1}-n_p}\right).$$

Ceci étant positif d'après (6) pour p grand, nous arriverons à la conclusion voulue, en supprimant, si nécessaire, quelques termes au début de la suite $\{n_p\}$ et en conservant la même notation. (C. q. f. d.)

Nous ne savons pas compléter cette proposition par son inverse, tandis que nous avons seulement la suivante :

V. *Soient* $\underline{g}(x)$, $\underline{g}'(x)$, $\underline{h}(y)$, $\underline{k}(y)$ *de même nature que* $g(x)$, $g'(x)$, $h(y)$, $k(y)$. *S'il y a une suite croissante des entiers* $n_1, n_2,\ldots, n_p,\ldots$ *tels que*

$$|a_{n_p}|e^{k(n_p)}>1,\quad \frac{\displaystyle\int_{\frac{1}{n_{p+1}}}^{\frac{1}{n_p}}g\left[h\left(\frac{1}{t}\right)\right]dt}{\dfrac{1}{n_p}-\dfrac{1}{n_{p+1}}}>\underline{g}\left(\frac{\displaystyle\int_{n_p}^{n_{p+1}}h(t)dt}{n_{p+1}-n_p}\right),$$

on a alors, pour x *assez voisin de* X,

$$m(e^x)>e^{\underline{q}(x)}.$$

Dém.—Il suffit de prouver que l'on a, pour x assez voisin de X,

$$\frac{|a_{n_p}|e^{n_p x}}{e^{q(x)}}>1,$$

p étant une fonction convenable de x. En tenant compte de (3) nous avons, par hypothèse,

$$\frac{|a_{n_p}|e^{n_p x}}{e^{q(x)}}>e^{-k(n_p)+\underline{k}(n_p)-\int_y^{n_p}[\underline{h}(t)-\underline{h}(y)]dt}=e^{\underline{\Theta}(n_p,\ y)}\quad[y=\underline{g}'(x)],$$

où l'on pose

$$\underline{\Theta}(u,\ y)=-k(u)+\underline{k}(u)-\int_y^u[\underline{h}(t)-\underline{h}(y)]dt.$$

Or, $\underline{\Theta}(n_p,\ y)$ est, comme fonction de y, maximum pour $y=n_p$ et elle décroit autant que y s'éloigne de n_p ; donc, si l'on prend des nombres y_p ($p=1,\ 2,\ldots$) comme dans la démonstration de IV, tels que

$$y_p=\underline{g}'\left(\frac{\displaystyle\int_{n_p}^{n_{p+1}}h(t)dt}{n_{p+1}-n_p}\right),\quad \underline{\Theta}(n_p,\ y_p)=\underline{\Theta}(n_{p+1},\ y_p)\quad[p=1,\ 2,\ldots],$$

et si l'on définit ϕ comme fonction de y de façon que $y_{p-1}<y\leqq y_p$, alors on a

$$\underline{\Theta}(n_p,\ y)\geqq\underline{\Theta}(n_p,\ y_{p-1}) \text{ ou } \underline{\Theta}(n_p,\ y_p)$$

suivant que y est entre y_{p-1} et n_p ou entre n_p et y_p. Ainsi, ϕ étant une fonction de y, indéfiniment croissante, il suffit de prouver que

$$\underline{\Theta}(n_p,\ y_{p-1})=\underline{\Theta}(n_{p-1},\ y_{p-1})>0 \text{ et } \underline{\Theta}(n_p,\ y_p)>0\ ;$$

ces deux conditions sont identiques et elles reviennent, comme dans la démonstration de IV, à

$$\underline{\Theta}(n_p,\ y_p)=\frac{\int_{\frac{1}{n_{p+1}}}^{\frac{1}{n_p}} g\left[h\left(\frac{1}{t}\right)\right]dt}{\frac{1}{n_p}-\frac{1}{n_{p+1}}}-\underline{g}\left(\frac{\int_{n_p}^{n_{p+1}}h(t)dt}{n_{p+1}-n_p}\right)>0\ ;$$

c'est ce qui a lieu par hypothèse. (C. q. f. d.)

Pour déduire de ces propositions la condition pour que $\lim_{x\to X}\frac{m(e^x)}{e^{\varphi(x)}}=1$, nous ferons la remarque suivante :

VI. *La fonction*

$$\chi(\alpha,\ \beta)=\frac{\int_{\frac{1}{\beta}}^{\frac{1}{\alpha}} g\left[h\left(\frac{1}{t}\right)\right]dt}{\frac{1}{\alpha}-\frac{1}{\beta}}-g\left(\frac{\int_{\alpha}^{\beta}h(t)dt}{\beta-\alpha}\right)\quad[g'(X_0)\leqq\alpha<\beta<+\infty]$$

est toujours négative ; elle est croissante par rapport à une variable α $[g'(X_0)\leqq\alpha<\beta]$ *et décroissante par rapport à* β $[\alpha<\beta<+\infty]$; *et* $\lim_{\substack{\alpha\to A\\ \beta\to A\\ (\alpha<\beta)}}\chi(\alpha,\ \beta)=0$, A *étant un nombre arbitraire tel que* $g'(X_0)\leqq A<+\infty$.

Dém.—Comme dans la démonstration de IV, on a

$$\chi(\alpha,\ \beta)=-\int_y^{\alpha}[h(t)-h(y)]dt=-\int_y^{\beta}[h(t)-h(y)]dt,$$

y étant uniquement déterminé par cette égalité de sorte que $\alpha<y<\beta$; de cette propriété on déduira immédiatement toutes les conclusions.

(C. q. f. d.)[1]

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1. Il est aussi à remarquer que, en posant $\xi=\frac{\int_{\alpha}^{\beta}h(t)dt}{\beta-\alpha}-h(\alpha)$ $[g'(X_0)\leqq\alpha<\beta]$, on a par (2),

$$\frac{\int_{\frac{1}{\beta}}^{\frac{1}{\alpha}} g\left[h\left(\frac{1}{t}\right)\right]dt}{\frac{1}{\alpha}-\frac{1}{\beta}}=\frac{\frac{k(\beta)}{\beta}-\frac{k(\alpha)}{\alpha}}{\frac{1}{\alpha}-\frac{1}{\beta}}=g[h(\alpha)]+\alpha\xi,\ g\left(\frac{\int_{\alpha}^{\beta}h(t)dt}{\beta-\alpha}\right)=g[h(\alpha)+\xi].\quad *$$

VII. *Dans le cas où* $\lim_{r\to R} N(r)=+\infty$, *pour que* $\lim_{x\to X}\frac{m(e^x)}{e^{g(x)}}=1$, *il faut et il suffit que*

$$\overline{\lim_{n\to\infty}}\,|a_n|\,e^{k(n)}=1$$

et qu'il y ait une suite $\{n_p\}$ *de manière que*

$$\lim_{p\to\infty}|a_{n_p}|\,e^{k(n_p)}=1,\quad \lim_{p\to\infty}\chi(n_p,\ n_{p+1})=0.$$

Dém.—D'après III, pour que $\overline{\lim_{x\to X}}\frac{m(e^x)}{e^{g(x)}}=1$, il faut et il suffit que

$$\overline{\lim_{n\to\infty}}\,|a_n|\,e^{k(n)}=1.$$

D'autre part, pour que $e^{g(x)+c}>m(e^x)>e^{g(x)-c}$ $(c>0)$ pour x suffisamment approché de X, il est nécessaire qu'il y ait une suite $\{n_p\}$ telle que

$$|a_{n_p}|\,e^{k(n_p)+c}>1,\quad \frac{\int_{\frac{1}{n_{p+1}}}^{\frac{1}{n_p}}\left\{g\left[h\left(\frac{1}{t}\right)\right]+c\right\}dt}{\frac{1}{n_p}-\frac{1}{n_{p+1}}}>g\left(\frac{\int_{n_p}^{n_{p+1}}h(t)\,dt}{n_{p+1}-n_p}\right)-c\,;$$

c'est une conséquence de IV, en prenant X_0 assez voisin de X et en posant $g(x)+c$, $g(x)-c$ au lieu de $\bar g(x)$, $g(x)$. Par suite en faisant $c\to 0$, nous vérifierons la nécessité des conditions énoncées, en vertu de VI.

Réciproquement, si ces conditions sont remplies, nous avons, d'après V, $m(e^x)>e^{g(x)-2c}$ $(c>0)$ pour x assez voisin de X, car, en posant $g(x)-c$, $g(x)-2c$ au lieu de $g(x)$, $\underline{g}(x)$ de V, nous avons à satisfaire aux conditions :

$$|a_{n_p}|\,e^{k(n_p)+c}>1,\quad \frac{\int_{\frac{1}{n_{p+1}}}^{\frac{1}{n_p}}\left\{g\left[h\left(\frac{1}{t}\right)\right]-c\right\}dt}{\frac{1}{n_p}-\frac{1}{n_{p+1}}}>g\left(\frac{\int_{n_p}^{n_{p+1}}h(t)\,dt}{n_{p+1}-n_p}\right)-2c,$$

* On obtient autres égalités analogues en échangeant α et β. Ces égalités nous permet de conclure VI. Elles nous montre aussi que *la fonction*

$$\frac{\int_{\frac{1}{\beta}}^{\frac{1}{\alpha}} g\left[h\left(\frac{1}{t}\right)\right]dt}{\frac{1}{\alpha}-\frac{1}{\beta}}:\ g\left(\frac{\int_{\alpha}^{\beta}h(t)\,dt}{\beta-\alpha}\right)\quad [0<\alpha<\beta<+\infty,\ g[h(\alpha)]>0]$$

est décroissante par rapport à β *et que, si* $\lim_{x\to X}g(x)=+\infty$, *elle tend vers* 0 *pour* $\beta\to+\infty$.

lesquelles sont satisfaites par hypothèse. Ici c étant arbitraire, il arrive que $\lim_{x\to X}\frac{m(e^x)}{e^{g(x)}}=1$. (C. q. f. d.)

VIII. *Étant donné une fonction $g(x)$ définie comme ci-dessus, pour qu'il existe une série entière $f(z)$ telle que $\lim_{x\to X}\frac{m(e^x)}{e^{g(x)}}=1$, il faut et il suffit que*

$$\lim_{n\to\infty}\chi(n,\,n+1)=0.$$

Dém.—Lorsque $\lim_{x\to X}g(x)<+\infty$, cette proposition est triviale ; au contraire, si $\lim_{x\to X}g(x)=+\infty$, on a nécessairement $\lim_{r\to R}M(r)=+\infty$, et, en se reportant alors à VI et VII, on prouve cette proposition. (C. q. f. d.)

Exemple.—*Il ne peut exister une fonction entière pour laquelle*

$$M(r)=r^{\log\log r+o\left(\frac{1}{(\log\log r)^2}\right)}\quad(r\to\infty).$$

Dém.—S'il existait une telle fonction, en posant $r'=2r$ aux inégalités (1), on aurait

$$m(r)=r^{\log\log r+o\left(\frac{1}{(\log\log r)^2}\right)},$$

donc, si nous prenons $g(x)$ telle que $\lim_{x\to X}\frac{m(e^x)}{e^{g(x)}}=1$, nous aurions

$$g(x)=x\log x+o\left(\frac{x}{(\log x)^2}\right).$$

On aurait alors

$$g'(x)=\log x+1+\varepsilon(x)\ \text{ avec }\ \varepsilon(x)=o\left(\frac{1}{\log x}\right);$$

parce que, pour x_0 et x_1 grands $(x_0<x_1)$,

$$(x_1-x_0)\,[\log x_0+1+\varepsilon(x_0)]<g(x_1)-g(x_0)<(x_1-x_0)\,[\log x_1+1+\varepsilon(x_1)],$$

$$g(x_1)-g(x_0)=\int_{x_0}^{x_1}(\log t+1)\,dt+o\left(\frac{x_1}{(\log x_1)^2}\right),$$

dont la première inégalité entraîne

$$(x_1-x_0)\varepsilon(x_0)<\int_{x_0}^{x_1}\log\frac{t}{x_0}\,dt+o\left(\frac{x_1}{(\log x_1)^2}\right),$$

d'où, en posant $x_1=x_0\left(1+\frac{\eta(x_0)}{\log x_0}\right)$ où $\eta(x)$ est une fonction convenable telle que $0<\eta(x)=o(1)$, on obtient

$$\varepsilon(x_0)<\frac{\eta(x_0)}{\log x_0}+\frac{o(1)}{\eta(x_0)\log x_0}=o\left(\frac{1}{\log x_0}\right);$$

et la seconde inégalité entraîne, de même,

$$\varepsilon(x) > o\left(\frac{1}{\log x}\right).$$

Cela posé, de $y = g'(x)$ nous tirerons

$$x = h(y) = e^{y-1-\varepsilon(h(y))} = e^{y-1}\left[1 + o\left(\frac{1}{y}\right)\right].$$

Nous avons enfin à examiner la condition $\lim_{n\to\infty} \chi(n, n+1) = 0$ de VIII. D'abord

$$\int_{\frac{1}{n+1}}^{\frac{1}{n}} g\left[h\left(\frac{1}{t}\right)\right] dt = \frac{\int_{\frac{1}{n+1}}^{\frac{1}{n}} e^{t-1}\left(\frac{1}{t} - 1\right) dt}{\frac{1}{n} - \frac{1}{n+1}} \left[1 + o\left(\frac{1}{n}\right)\right],$$

où

$$\frac{\int_{\frac{1}{n+1}}^{\frac{1}{n}} e^{t-1}\left(\frac{1}{t} - 1\right) dt}{\frac{1}{n} - \frac{1}{n+1}} = n(n+1)\int_n^{n+1} e^{u-1}\left(\frac{1}{u} - \frac{1}{u^2}\right) du > \frac{n^2}{n+1}\int_n^{n+1} e^{u-1} du$$

$$= \left[n - 1 + o\left(\frac{1}{n}\right)\right](e^n - e^{n-1}).$$

Puis

$$g\left(\int_n^{n+1} h(t)\, dt\right) = (e^n - e^{n-1}) \log (e^n - e^{n-1}) \cdot \left[1 + o\left(\frac{1}{n}\right)\right]$$

$$= (e^n - e^{n-1})\,[n - 1 + \log (e-1)]\left[1 + o\left(\frac{1}{n}\right)\right].$$

Par suite, on a $\lim_{n\to\infty} \chi(n, n+1) = -\infty$; ce qui est en désaccord avec la condition de VIII. (C. q. f. d.)

§2.—Relations entre la croissance de $M(r)$ et celle de $\{|a_n|\}$.

Exemple 1.—Séries entières à rayon de convergence fini.

Supposons que $R = 1$ et comparons $\log M(r)$ à $\frac{b}{(1-r)^a}$ où a et b sont des paramètres positifs.

Voici une préliminaire :

1. $$\overline{\lim_{r\to 1-0}} \frac{\log M(r)}{(1-r)^{-a}} = \overline{\lim_{r\to 1-0}} \frac{\log m(r)}{(1-r)^{-a}},\quad \lim_{r\to 1-0} \frac{\log M(r)}{(1-r)^{-a}} = \lim_{r\to 1-0} \frac{\log m(r)}{(1-r)^{-a}}.$$

Dém.—En posant $r' = r + (1-r)^2$ dans (1), on a de suite ces égalités. (C. q. f. d.)

Cela posé, nous obtenons par III une relation entre $M(r)$ et $\{|a_n|\}$:

ii.[1] *Soit* $$\overline{\lim_{r\to 1-0}}\frac{\log M(r)}{(1-r)^{-\alpha}}=\varkappa.$$

Alors $$\overline{\lim_{n\to\infty}}\frac{\log|a_n|}{\left(\frac{n}{a}\right)^{\frac{\alpha}{\alpha+1}}}=\varkappa^{\frac{1}{\alpha+1}}(\alpha+1).$$

Dém.—Par i, $\varkappa$ est borne inférieure de nombres positifs b tels que

$$\lim_{x\to-0}\frac{m(e^x)}{e^{b(-x)^{-\alpha}}}=0;$$

donc, d'après III, en posant

$$g(x)=b(-x)^{-\alpha},\ g'(x)=b\alpha(-x)^{-(\alpha+1)},\ h(y)=-\left(\frac{b\alpha}{y}\right)^{\frac{1}{\alpha+1}},$$

$$k(y)=yh(y)-g[h(y)]=-(b\alpha)^{\frac{1}{\alpha+1}}y^{\frac{\alpha}{\alpha+1}}-b\left(\frac{b\alpha}{y}\right)^{-\frac{\alpha}{\alpha+1}}=-b^{\frac{1}{\alpha+1}}(\alpha+1)\left(\frac{y}{\alpha}\right)^{\frac{\alpha}{\alpha+1}},$$

$\varkappa$ est borne inférieure de b tels que

$$\lim_{n\to\infty}|a_n|e^{-b^{\frac{1}{\alpha+1}}(\alpha+1)\left(\frac{n}{\alpha}\right)^{\frac{\alpha}{\alpha+1}}}=0,$$

d'où découle le résultat énoncé. (C. q. f. d.)

De cette proposition on obtient immédiatement la suivante donnée par M. Beuermann *loc. cit.*

Soit $$\overline{\lim_{r\to 1-0}}\frac{\overset{+}{\log}\overset{+}{\log}M(r)}{\log\frac{1}{1-r}}=\mu.$$

Alors $$\overline{\lim_{n\to\infty}}\frac{\overset{+}{\log}\overset{+}{\log}|a_n|}{\log n}=\frac{\mu}{\mu+1}.$$

Maintenant nous allons appliquer les règles IV et V.

iii. *En posant* $$\lim_{r\to 1-0}\frac{\log M(r)}{(1-r)^{-\alpha}}=\varkappa_1,$$

si $+\infty>\varkappa\geqq\varkappa_1>0$, *on peut déterminer une suite croissante des entiers* $\{n_p\}$ *pour laquelle*

$$\lim_{p\to\infty}\frac{\log|a_{n_p}|}{\left(\frac{n_p}{a}\right)^{\frac{\alpha}{\alpha+1}}}\geqq\varkappa_1^{\frac{1}{\alpha+1}}(\alpha+1)\ \textit{et}\ \overline{\lim_{p\to\infty}}\frac{n_{p+1}}{n_p}\leqq\lambda$$

où λ *est la racine de l'équation*

$$\frac{\int_{\frac{1}{\lambda}}^{1}\left(\frac{1}{\tau}\right)^{\frac{\alpha}{\alpha+1}}d\tau}{1-\frac{1}{\lambda}}:\left[\frac{\int_{1}^{\lambda}\left(\frac{1}{\tau}\right)^{\frac{1}{\alpha+1}}d\tau}{\lambda-1}\right]^{-\alpha}=\frac{\varkappa_1}{\varkappa}\quad(\lambda\geqq 1).$$

1. Fujiwara, *loc. cit.*, p. 220.

Remarquons que le premier membre de cette dernière équation est, comme fonction de λ, continu et décroissant, et qu'il varie de 1 à 0 d'après la note de la p. 258 en faisant $g(x)=(-x)^{-\alpha}$, $\alpha=1$, $\beta=\lambda$; et que par suite λ est uniquement déterminé.

Dém.—En posant dans IV,

$$\bar{g}(x)=b(-x)^{-\alpha}\ (b>\varkappa),\ g(x)=b_1(-x)^{-\alpha}\ (b_1<\varkappa_1),$$

on a, pour une suite $\{n_p\}$,

$$|a_{n_p}|\,e^{-b_1^{\frac{1}{\alpha+1}}(\alpha+1)\left(\frac{n_p}{a}\right)^{\frac{\alpha}{\alpha+1}}}>1,\quad \frac{\displaystyle\int_{\frac{1}{n_{p+1}}}^{\frac{1}{n_p}} b\left(\frac{1}{bat}\right)^{\frac{\alpha}{\alpha+1}}dt}{\dfrac{1}{n_p}-\dfrac{1}{n_{p+1}}}>b_1\left[\frac{\displaystyle\int_{n_p}^{n_{p+1}}\left(\frac{ba}{t}\right)^{\frac{1}{\alpha+1}}dt}{n_{p+1}-n_p}\right]^{-\alpha}.$$

La première de ces inégalités donne

$$\frac{\log|a_{n_p}|}{\left(\dfrac{n_p}{a}\right)^{\frac{\alpha}{\alpha+1}}}>b_1^{\frac{1}{\alpha+1}}(\alpha+1),$$

et la deuxième donne, en transformant la variable d'intégration,

$$-\frac{b\displaystyle\int_{\frac{n_p}{n_{p+1}}}^{1}\left(\frac{1}{\tau}\right)^{\frac{\alpha}{\alpha+1}}d\tau}{1-\dfrac{n_p}{n_{p+1}}}>b_1\left(\frac{\displaystyle\int_1^{\frac{n_{p+1}}{n_p}}\left(\frac{1}{\tau}\right)^{\frac{1}{\alpha+1}}d\tau}{\dfrac{n_{p+1}}{n_p}-1}\right)^{-\alpha}.$$

En prenant b et b_1 plus approchés de $\varkappa$ et $\varkappa_1$, on obtient une suite correspondante $\{n_p\}$, et en tirant convenablement des entiers de ces suites $\{n_p\}$, et en les désignant de nouveau par $n_p(p=1,2,\ldots)$, on peut leur faire satisfaire aux conditions indiquées. (C. q. f. d.)

iv. *Si l'on trouve une suite* $\{n_p\}$ *telle que*

$$\overline{\lim_{p\to\infty}}\frac{\log|a_{n_p}|}{\left(\dfrac{n_p}{a}\right)^{\frac{\alpha}{\alpha+1}}}\geqq\varkappa'^{\frac{1}{\alpha+1}}(\alpha+1)>0\ \textit{et}\ \overline{\lim_{p\to\infty}}\frac{n_{p+1}}{n_p}=\lambda<+\infty,$$

alors

$$\frac{\displaystyle\int_{\frac{1}{\lambda}}^{1}\left(\frac{1}{\tau}\right)^{\frac{\alpha}{\alpha+1}}d\tau}{1-\dfrac{1}{\lambda}}:\left(\frac{\displaystyle\int_1^{\lambda}\left(\frac{1}{\tau}\right)^{\frac{1}{\alpha+1}}d\tau}{\lambda-1}\right)^{-\alpha}\leqq\frac{\varkappa_1}{\varkappa'}.$$

Dém.—En posant dans V,

$$g(x)=b'(-x)^{-\alpha}\ (b'<\varkappa'),\ g(x)=b_1(-x)^{-\alpha},$$

et en faisant tendre b' vers $\varkappa'$, on prouve ce résultat en vertu de i. (C. q. f. d.)

Des propositions iii et iv, nous obtenons évidemment les deux autres:

v. *Pour que* $\varkappa$ *et* $\varkappa_1$ *soient finis et positifs tous les deux, il faut*

et il suffit que la condition de ii *étant réalisée, on puisse trouver une suite* $\{n_p\}$ *telle que*

$$\varlimsup_{p\to\infty}\frac{\log|a_{n_p}|}{n_p^{\frac{a}{a+1}}}>0 \quad et \quad \varliminf_{p\to\infty}\frac{n_{p+1}}{n_p}<+\infty.$$

vi.[1] *Pour que* $\varkappa$ *et* $\varkappa_1$ *soient finis et positifs et* $\varkappa=\varkappa_1$, *il faut et il suffit que la condition de* ii *étant réalisée, on puisse trouver une suite* $\{n_p\}$ *telle que*

$$\lim_{p\to\infty}\frac{\log|a_{n_p}|}{\left(\frac{n_p}{a}\right)^{\frac{a}{a+1}}}=\varkappa^{\frac{1}{a+1}}(a+1) \quad et \quad \lim_{p\to\infty}\frac{n_{p+1}}{n_p}=1.$$

M. Fujiwara *loc. cit.* montre aussi que, *en posant*

$$\varlimsup_{r\to 1-0}\frac{\log M(r)}{\log\frac{1}{1-r}}=\omega, \quad \varlimsup_{n\to\infty}\frac{\log|a_n|}{\log n}=\mu,$$

on a

$$\mu\leqq\omega\leqq\mu+1.$$

C'est ce que l'on obtient en appliquant notre théorie. Mais M. Fujiwara déduit, de plus, en utilisant l'inégalité connue

$$\sum_{n=0}^{\infty}|a_n|^2r^{2n}\leqq M(r)^2,$$

que, *si* $\lim\frac{\log|a_n|}{\log n}$ *existe et égale à* μ, *on a* $\varlimsup_{r\to 1-0}\frac{\log M(r)}{\log\frac{1}{1-r}}\geqq\mu+\frac{1}{2}$. De tels résultats sont hors de la portée de notre théorie, parce que nous nous appuyons seulement sur l'inégalité $m(r)\leqq M(r)$.

En outre M. Hiong *loc. cit.* donne quelques résultats généraux concernant le cas où la croissance est plus rapide que jusqu'ici. On pourrait les obtenir par notre méthode, d'une façon analogue au cas des fonctions entières considéré par M. Hiong, que nous nous occuperons de refaire à titre d'Exemple 2.

Exemple 2.—Fonctions entières.

Supposont que $R=+\infty$ et comparons $\log M(r)$ à $U(r)^c$ où, c étant un paramètre positif, $U(r)$ *est une fonction à croissance normale* d'après M. Hiong, de sorte que $U(r)$ est une fonction continue de r $[r_0\leqq r<+\infty]$, croissante[3], tendant vers $+\infty$ pour $r\to+\infty$, et telle que

1. Valiron, Mémoire cité, p. 54–55.

2. M. Fujiwara écrit $\varlimsup$ au lieu de $\varlimsup$, mais le raisonnement est tout à fait valable au cas dernier.

3. M. Hiong suppose que $U(r)$ soit non décroissante, mais cela ne diffère pas essentiellement.

$$U\left(r+\frac{r}{\log U(r)}\right)<U(r)^{1+\varepsilon(r)} \text{ où } \lim_{r\to\infty}\varepsilon(r)=0.$$

Nous supposerons, de plus, que la croissance de $U(r)$ soit assez rapide de sorte que $\lim_{r\to\infty}\frac{\log r}{U(r)^{\varepsilon}}=0$, quel petit que soit un nombre positif ε[1].

Commençons par les préliminaires :

vii. $$\varliminf_{r\to\infty}\frac{\log^{+}\log^{+}M(r)}{\log U(r)}=\varliminf_{r\to\infty}\frac{\log^{+}\log^{+}m(r)}{\log U(r)},$$

$$\varlimsup_{r\to\infty}\frac{\log^{+}\log^{+}M(r)}{\log U(r)}=\varlimsup_{r\to\infty}\frac{\log^{+}\log^{+}m(r)}{\log U(r)}.$$

Dém.—En posant $r'=r+\frac{r}{\log U(r)}$ dans (1), on a de suite ces égalités. (C. q. f. d.)

viii. *Si nous posons* $g_c(x)=\int U(e^x)^c dx$, $g_c(x)$ *possède les propriétés caractérisant* $g(x)$ *et l'on a, pour* x *grand,*

$$g_c(x)=U(e^x)^{c+\varepsilon(x)} \text{ avec } \lim_{x\to\infty}\varepsilon(x)=0.$$

Dém.—On a $g_c'(x)=U(e^x)^c$; c'est une fonction continue croissante. Or, $g_c(x)>(x-x')U(e^{x'})^c \quad (x'<x)$,

d'où, en posant

$$x'=x-\log\left(1+\frac{1}{\log U(e^{x'})}\right),$$

on a

$$g_c(x)>\log\left(1+\frac{1}{\log U(e^{x'})}\right)\cdot U(e^{x'})^c>U(e^{x'})^{c'} \quad (c'<c),$$

et, en tenant compte de la condition de croissance normale, on obtient

$$g_c(x)>U(e^x)^{c''} \quad (c''<c');$$

d'autre part, on a

$$g_c(x)<xU(e^x)^c+\text{const.}<U(e^x)^{c'''} \quad (c'''>c).$$

c'' et c''' sont aussi approchés de c que l'on veut. (C. q. f. d.)

Les propositions vii et viii nous ramènent à l'application des résultats de §1. Or nous avons besoin du lemme suivant :

ix. $$\int_{y^{\gamma}}^{y}\log U^{-1}(t^{\frac{1}{c}})dt=y\log U^{-1}(y^{\frac{1-\varepsilon(y)}{c}}) \text{ avec } \lim_{y\to\infty}\varepsilon(y)=0,$$

où c *et* γ *sont des constantes,* $c>0$, $0<\gamma<1$, *et* $U^{-1}(y)$ *désigne la fonction inverse de* $y=U(x)$.

---

1. M. Hiong ne pose pas explicitement cette hypothèse, néanmoins il l'utilise dans les raisonnements.

Dém.—Soit ε un nombre positif petit tel que $\gamma<1-\varepsilon<1$. On a, pour y assez grand,

$$\int_{y^\gamma}^{y}\log U^{-1}(t^{\frac{1}{c}})dt<y\log U^{-1}(y^{\frac{1}{c}}),$$

et d'autre part,

$$\begin{aligned}\int_{y^\gamma}^{y}\log U^{-1}(t^{\frac{1}{c}})dt&>\int_{y^{1-\varepsilon}}^{y}\log U^{-1}(t^{\frac{1}{c}})dt\\&=(y-y^{1-\varepsilon})\log U^{-1}(y^{\frac{1-\varepsilon}{c}})+\int_{y^{1-\varepsilon}}^{y}\log\frac{U^{-1}(t^{\frac{1}{c}})}{U^{-1}(y^{\frac{1-\varepsilon}{c}})}dt\\&>(y-y^{1-\varepsilon})\log U^{-1}(y^{\frac{1-\varepsilon}{c}})+(y-y^{1-\varepsilon'})\log\frac{U^{-1}(y^{\frac{1-\varepsilon'}{c}})}{U^{-1}(y^{\frac{1-\varepsilon}{c}})}\\&\qquad(0<\varepsilon'<\varepsilon),\end{aligned}$$

où, en vertu de la condition de croissance normale,

$$\frac{U^{-1}(y^{\frac{1-\varepsilon'}{c}})}{U^{-1}(y^{\frac{1-\varepsilon}{c}})}>1+\frac{1}{\log y^{\frac{1-\varepsilon}{c}}},$$

donc

$$(y-y^{1-\varepsilon'})\log\frac{U^{-1}(y^{\frac{1-\varepsilon'}{c}})}{U^{-1}(y^{\frac{1-\varepsilon}{c}})}>(y-y^{1-\varepsilon'})\frac{1+o(1)}{\log y^{\frac{1-\varepsilon}{c}}}>y^{1-\varepsilon''}\quad(0<\varepsilon''<\varepsilon),$$

qui est, d'après hypothèse, supérieur à

$$y^{1-\varepsilon}\log U^{-1}(y^{\frac{1-\varepsilon}{c}});$$

par conséquent

$$\int_{y^\gamma}^{y}\log U^{-1}(t^{\frac{1}{c}})dt>y\log U^{-1}(y^{\frac{1-\varepsilon}{c}}).\qquad\text{(C. q. f. d.)}$$

Maintenant nous arriverons par III à une relation de M. Hiong.

x. *Soit*
$$\overline{\lim_{r\to\infty}}\frac{\log^{+}\log^{+}M(r)}{\log U(r)}=\rho.$$

Alors
$$\rho=\overline{\lim_{n\to\infty}}\frac{\log n}{\log U\left(\frac{1}{\sqrt[n]{|a_n|}}\right)}.$$

Dém.—D'après vii et viii, ρ est borne inférieur de nombres positifs c tels que $\overline{\lim_{x\to\infty}}\frac{m(e^x)}{e^{g_c(x)}}=0$ (ou $<+\infty$), ou, d'après III, ρ est borne inférieur de c tel que $\overline{\lim_{n\to\infty}}|a_n|e^{k_c(n)}=0$ (ou $<+\infty$), $k_c(n)$ étant définie de même que $k(n)$. Or, nous avons

$$g_c(x)=\int U(e^x)^c dx,\ g_c'(x)=U(e^x)^c,\ h_c(y)=\log U^{-1}(y^{\frac{1}{c}}),$$

$$k_c(y)=\int\log U^{-1}(y^{\frac{1}{c}})dy,$$

et, par ix,

$$k_c(n)=n\log U^{-1}(n^{\frac{1-\eta(n)}{c}})+\text{const.},\ \lim_{n\to\infty}\eta(n)=0.$$

Par conséquent, ρ est borne inférieure de c tels que

$$\overline{\lim_{n\to\infty}}\,[\sqrt[n]{|a_n|}\,U^{-1}(n^{\frac{1}{c}})]^n=0\quad(\text{ou}<+\infty),$$

ou, ce qui revient au même,

$$\sqrt[n]{|a_n|}\,U^{-1}(n^{\frac{1}{c}})<1\ \text{pour}\ n>n_c.$$

Ceci est équivalent à la condition énoncée. (C. q. f. d.)

Cela posé, nous allons ensuite appliquer les règles IV et V.

xi. *En posant*

$$\lim_{r\to\infty}\frac{\overset{+}{\log}\overset{+}{\log}U(r)}{\log U(r)}=\rho_1,$$

si $+\infty>\rho\geqq\rho_1>0$, *on peut déterminer une suite croissante des entiers* $\{n_p\}$ *pour laquelle*

$$\lim_{p\to\infty}\frac{\log n_p}{\log U\left(\frac{1}{\sqrt[n_p]{|a_{n_p}|}}\right)}\geqq\rho_1\ \textit{et}\ \overline{\lim_{p\to\infty}}\frac{\log n_{p+1}}{\log n_p}\leqq\frac{\rho}{\rho_1}.$$

Dém.—En posant dans IV,

$$\bar g(x)=g_{\rho+\varepsilon}(x),\ g(x)=g_{\rho_1-\varepsilon}(x)\quad(\varepsilon>0),$$

on a, d'après vii, pour une suite $\{n_p\}$,

$$|a_{n_p}|\,e^{k\rho_1-\varepsilon(n_p)}>1,\ \frac{\displaystyle\int_{\frac{1}{n_{p+1}}}^{\frac{1}{n_p}} g_{\rho+\varepsilon}\left[h_{\rho+\varepsilon}\left(\frac{1}{t}\right)\right]dt}{\dfrac{1}{n_p}-\dfrac{1}{n_{p+1}}}>g_{\rho_1-\varepsilon}\left(\frac{\displaystyle\int_{n_p}^{n_{p+1}}h_{\rho+\varepsilon}(t)dt}{n_{p+1}-n_p}\right).$$

La première de ces inégalités donne, pour p assez grand,

$$|a_{n_p}|\,e^{n_p\log U^{-1}\left(n_p^{\frac{1}{\rho_1-\varepsilon}}\right)}>1\ \text{ou}\ \frac{\log n_p}{\log U\left(\frac{1}{\sqrt[n_p]{|a_{n_p}|}}\right)}>\rho_1-\varepsilon.$$

Quant à la deuxième inégalité, le premier membre devient par viii,

$$\frac{\displaystyle\int_{\frac{1}{n_{p+1}}}^{\frac{1}{n_p}}\left(\frac{1}{t}\right)^{1+\varepsilon(t)}dt}{\dfrac{1}{n_p}-\dfrac{1}{n_{p+1}}}\quad\text{où}\ \lim_{t\to0}\varepsilon(t)=0,$$

qui est, par intégration, égale à

$$n_p n_{p+1}^{\eta(p)}\ \text{où}\ \lim_{p\to\infty}\eta(p)=0\,;$$

et dans le second membre, prenant un nombre $\gamma(0<\gamma<1)$, on a pour $n_p\geqq n_{p+1}^{\gamma}$,

$$\frac{\int_{n_p}^{n_{p+1}} h_{\rho+\varepsilon}(t)dt}{n_{p+1}-n_p} > \log U^{-1}\left(n_{p+1}^{\frac{\gamma}{\rho+\varepsilon}}\right),$$

et, pour $n_p < n_{p+1}^{\gamma}$, d'après ix,

$$\frac{\int_{n_p}^{n_{p+1}} h_{\rho+\varepsilon}(t)dt}{n_{p+1}-n_p} > \frac{n_{p+1}\log U^{-1}\left(n_{p+1}^{\frac{1-\zeta(p)}{\rho+\varepsilon}}\right)}{n_{p+1}-n_p} > \log U^{-1}\left(n_{p+1}^{\frac{1-\zeta(p)}{\rho+\varepsilon}}\right) \text{ où } \lim_{p\to\infty}\zeta(p)=0,$$

donc, en faisant $\gamma\to 1$, on a pour tous les cas,

$$\frac{\int_{n_p}^{n_{p+1}} h_{\rho+\varepsilon}(t)dt}{n_{p+1}-n_p} > \log U^{-1}\left(n_{p+1}^{\frac{1-\theta(p)}{\rho+\varepsilon}}\right) \text{ avec } \lim_{p\to\infty}\theta(p)=0.$$

Nous avons ainsi par viii,

$$n_p n_{p+1}^{\eta(p)} > \left(n_{p+1}^{\frac{\rho_1-\varepsilon}{\rho+\varepsilon}}\right)^{1+\nu(p)} \text{ avec } \lim_{p\to\infty}\nu(p)=0\,;$$

par suite

$$\overline{\lim_{p\to\infty}}\frac{\log n_{p+1}}{\log n_p} \leqq \frac{\rho+\varepsilon}{\rho_1-\varepsilon}.$$

Pour chaque valeur de ε, on obtient une suite correspondante $\{n_p\}$, et, en tirant convenablement des entiers de ces suites $\{n_p\}$ et en les désignant de nouveau par $n_p(p=1, 2,\ldots)$, on peut les faire satisfaire aux conditions indiquées. (C. q. f. d.)

xii. *Si l'on trouve une suite* $\{n_p\}$ *telle que*

$$\underline{\lim_{p\to\infty}}\frac{\log n_p}{\log U\left(\frac{1}{\sqrt[n_p]{|a_{n_p}|}}\right)} = \rho' > 0 \text{ et } \overline{\lim_{p\to\infty}}\frac{\log n_{p+1}}{\log n_p} = \sigma < +\infty,$$

alors

$$\rho_1 \geqq \frac{\rho'}{\sigma}.$$

Dém.—Grâce à vii, viii et ix, on obtient ce résultat en appliquant V où l'on pose

$$g(x)=g_{\rho'-\varepsilon}(x), \quad \varphi(x)=g_{\frac{\rho'-\varepsilon}{\sigma+\varepsilon}}(x) \quad (\varepsilon>0),$$

et l'on fait tendre ε vers 0. (C. q. f. d.)

Des propositions xi et xii, nous concluons évidemment les deux autres :

xiii. *Pour que* ρ *et* ρ_1 *soient finis et positifs tous les deux, il faut et il suffit que la condition de* x *étant réalisée, on puisse trouver une suite* $\{n_p\}$ *telle que*

$$\underline{\lim_{p\to\infty}}\frac{\log n_p}{\log U\left(\frac{1}{\sqrt[n_p]{|a_{n_p}|}}\right)} > 0 \text{ et } \overline{\lim_{p\to\infty}}\frac{\log n_{p+1}}{\log n_p} < +\infty.$$

xiv. *Pour que* ρ *et* ρ_1 *soient finis et positifs et que* $\rho=\rho_1$, *il faut*

et il suffit que la condition de x *étant réalisée, on puisse trouver une suite* $\{n_p\}$ *telle que*

$$\lim_{p\to\infty}\frac{\log n_p}{\log U\left(\frac{1}{\sqrt[n_p]{|a_{n_p}|}}\right)}=\rho \text{ et } \lim_{p\to\infty}\frac{\log n_{p+1}}{\log n_p}=1.$$

xiv est du à M. Hiong.

Nous avons traité le cas $U(r)=r$ dans un mémoire antérieur[1].

On pourrait obtenir de même les autres résultats du mémoire cité[1], ainsi que divers résultats connus depuis longtemps, par exemple, ceux de M. Lindelöf[2].

1. Tôhoku Math. J. **38** (1933), p. 129.
2. Bull. Sci. Math. 2e série **27** (1903), 1re partie, p. 213.

Corrections à mon mémoire : Sur l'Unicité de la Solution de $\frac{dy}{dx}=f(x, y)$ [Ces Memoirs A **17** (1934). p. 319].

Page 320, ligne 7, et page 321, ligne 5 en remontant, *au lieu de* issue de O, *lire* qui passe par un point situé sur l'axe de x. (Sans cette correction, le Lemme de la page 321 ne serait pas exacte ; c'est ce que M. Hukuhara a bien voulu me signaler dans une lettre en juillet 1935.)

Page 324, ligne 6, *au lieu de* $[x_0-h\leqq x_i, x_i'\leqq x_0+h(i=1,2,\ldots,m)]$, *lire* $(x_0-h\leqq x_1\leqq x_1'\leqq x_2\leqq\ldots\ldots\leqq x_m\leqq x_m'\leqq x_0+h)$.

On the Anomalous Relation between the Specific Volume of Liquid Water and Temperature

By **Usaburo Yoshida**

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Abstract

By assuming that liquid water contains tiny ice-crystals having the size of the order of their unit cell, its specific volume, specific heat and coefficient of viscosity are inspected. A rough estimate of the amount of such ice-crystals in liquid water shows that it is approximately 0.6% at 0°C.

The presence of a minimum amount in a specific volume of water at 4°C is a strange characteristic of water differentiating it from other liquids whose specific volume increases gradually with the temperature. Water has another peculiarity in that its volume is increased by freezing it into ice. In the following some properties of water are examined by assuming that the above two peculiarities are related to each other. Though the condition of the aggregation of water molecules in the liquid state is not clear, if we assume that ideally they are not held very tightly and that their mutual positions can be displaced easily by thermal agitation, then the specific volume of water will gradually increase with temperature as is usually the case with other liquids and with water as well at the temperature range above 4°C. As the increase of the specific volume of water below 4°C can not be explained by this hypothesis, the writer assumes that some very tiny ice-crystals having the size of the order of their unit cell are present in liquid water even at a temperature above the freezing point: that is, that ordinary liquid water is a mixture of tiny fragments of ice and of ideal liquid water so constituted that the relative positions of its molecules can be moved easily by thermal agitation. Though the existence of each one of such ice-crystals in ordinary liquid water is very unstable, forming and disappearing momentarily here and there, yet a certain fraction of one gramme of ordinary liquid water, which fraction is determined by the temperature, will remain in the frozen state. Let the value of this fraction at the temperature t° C be x, and assume that the manner of the decrease of x with the temperature is expressed roughly by an exponential function, then x is

$$x = x_0 e^{-\alpha t} \quad \text{.....(1)},$$

where x_0 is the total mass of the ice fragments in one gramme of liquid water at 0°C.

Let the functional form representing the relation between the specific volume of ideal liquid water and the temperature be $f(t)$, and the specific volume of ice at 0°C be 1.0917; then, by neglecting the change of the specific volume of the ice fragments according to the temperature, the specific volume V_s of ordinary liquid water at the temperature t°C is

$$\begin{aligned} V_s &= 1.0917x + (1-x)f(t) = f(t) + \{1.0917 - f(t)\}x \\ &= f(t) + \{1.0917 - f(t)\}x_0 e^{-\alpha t} \quad \text{.....(2)}. \end{aligned}$$

The value of $f(t)$ in the bracket ought not be known very accurately in the present case, because the value of x_0 is estimated to be less than 0.01 (as will be seen later). By giving this term a constant value of 0.9998 we obtain

$$V_s = f(t) + 0.0919 x_0 e^{-\alpha t} \quad \text{.....(3)}.$$

When the temperature increases from 0°C, the specific volume of water V_s increases by the thermal expansion of the ideal liquid water, and decreases at the same time by the melting of some ice fragments. Though the occurrence of the minimum specific volume of water at 4°C can be understood qualitatively from the above two causes, the presence of the unknown term $f(t)$ in the above equation does not allow us to make even a rough estimation of the values of x_0 and α.

If the above is correct the specific heat of water must also be influenced by the melting of some part of the suspended ice fragments when its temperature is raised. According to Callender[1] the specific heat of water takes a minimum value at 35°—40°C. But with ideal liquid water, it seems natural to suppose that its specific heat represented by a functional form of $F(t)$ increases gradually with the temperature t as in the case of the specific volume. Let the specific heat of ice and the latent heat of fusion per gramme of ice be 0.5 and 80 calories respectively, then from the above consideration the specific heat c of ordinary liquid water at the temperature t becomes

$$\begin{aligned} c &= 0.5x + (1-x)F(t) + 80\alpha x \\ &= F(t) + \{80\alpha + 0.5 - F(t)\}x_0 e^{-\alpha t}. \end{aligned}$$

If we take the value of $F(t)$ in the bracket as approximately 1, the above expression becomes

1. Landolt-Börnstein: Physikalisch-chemische Tabellen, p. 1250 (1923)

$$c = F(t) + \{80\alpha - 0.5\} x_0 e^{-\alpha t} \quad \text{.....(4).}$$

By putting $c - F(t) = A$, $\{80\alpha - 0.5\} x_0 = B$, equation (4) becomes

$$A = Be^{-\alpha t} \quad \text{.....(5).}$$

If we put $V_s - f(t) = E$, $0.0919 x_0 = F$, equation (3) can be written in a similar form

$$E = Fe^{-\alpha t} \quad \text{.....(6).}$$

Equations (5) and (6) are very simple, and we obtain in logarithmic forms

$$\log_{10} A = \log_{10} B - \alpha t \log_{10} e \quad \text{.....(7).}$$

$$\log_{10} E = \log_{10} F - \alpha t \log_{10} e \quad \text{.....(8).}$$

These equations show that $\log_{10} A$ and $\log_{10} E$ decrease linearly with t having the same inclination. In order to get the values of A and E we must know the values of $F(t)$ and $f(t)$; and this is done experimentally by exterpolating the specific heat—temperature curve and the specific volume—temperature curve from the higher temperature range, so that the values of $F(t)$ and $f(t)$ asymptotically approach the actual observations of specific heat and specific volume respectively at the higher temperature range, and always decrease with decreasing temperatures. From these values of $F(t)$ and $f(t)$ those of A and E are immediately obtained by subtracting the values of $F(t)$ and $f(t)$ from the observed values of c and V_s respectively for various values of t. The values of $F(t)$ and $f(t)$ must be so chosen that $\log_{10} A$ and $\log_{10} E$ decrease linearly with t having the same inclination; and this is done by repeating differently several times the manner of exterpolation as stated above. Next the values of B and F are obtained from the graphs showing the linear relations between t and $\log_{10} A$ and $\log_{10} E$ as represented by (7) and (8). The value of x_0 obtained from the values of B and F thus found must be the same if the values of $F(t)$ and $f(t)$ are correct. This condition is almost satisfied by repeated trials at finding the values of $F(t)$ and $f(t)$. After such efforts the writer finally accepted the following values of x_0 and α as the best estimate.

$$x_0 = 0.006, \quad \alpha = 0.0644.$$

From these values of x_0 and α, the total mass of the tiny ice crystals supposed to be suspended in one gramme of ordinary liquid water at the temperature t°C becomes

$$x = 0.006 e^{-0.0644t} \quad \text{.....(9).}$$

By using the values of x calculated with (9) for various values of temperature t, $f(t)$ and $F(t)$ are recalculated by means of the equations (3) and (4), and they are represented respectively by the curves drawn

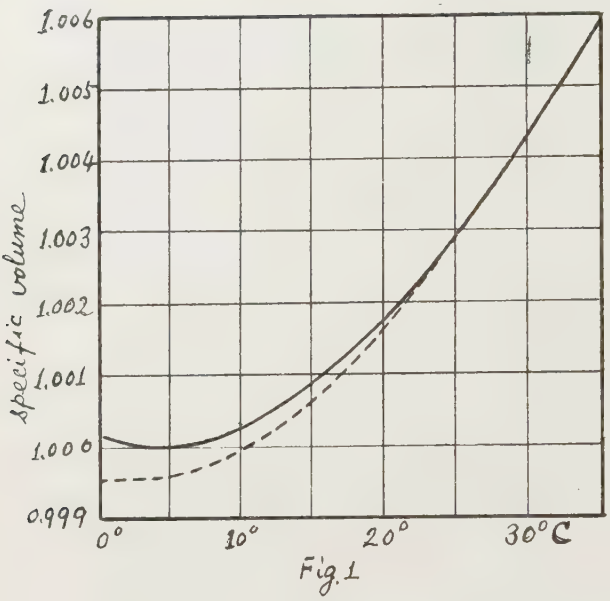


Fig. 1

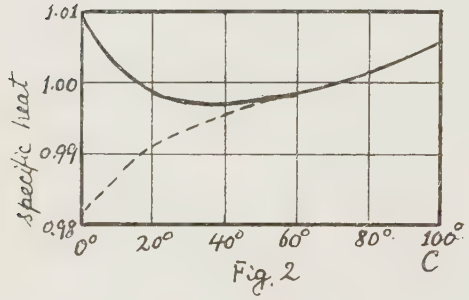


Fig. 2

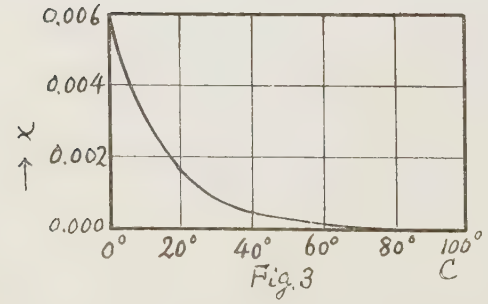


Fig. 3

in broken lines in Figs. 1 and 2. The conditions that the specific volume and the specific heat of ideal liquid water increase gradually with increasing temperature, and that they approach asymptotically at the higher temperature range those actually observed with ordinary water respectively seem to be almost satisfied by these two curves in Figs. 1 and 2, where the curves drawn in full lines show the values of the specific volume and the specific heat of ordinary liquid water actually observed. This fact seems to indicate conversely that the above estimate of the amount of tiny ice crystals suspended in liquid water is not inadequate. The amount of floating tiny ice crystals calculated with (9) for various temperatures is shown by the curve in Fig. 3.

The specific volume of ice is larger than that of water, and consequently when the pressure increases by one atmosphere the melting point of ice decreases by about 0.007°C. Thus if ordinary water contains some quantity of ice, as is supposed

above, the amount of ice at a certain temperature must become less when the pressure is increased. Consequently when a sufficiently high pressure is applied to water, the ice crystals suspended in it will almost disappear even at a temperature very near to 0°C, resulting in the disappearance of the minimum in the specific volume—temperature curve. This point is ascertained experimentally by Bridgman[1] by raising the pressure above 2000Kg/cm². Three curves obtained with the pressures of one atmosphere, 1000 Kg/cm², and 2000 Kg/cm² are shown in Fig. 4.

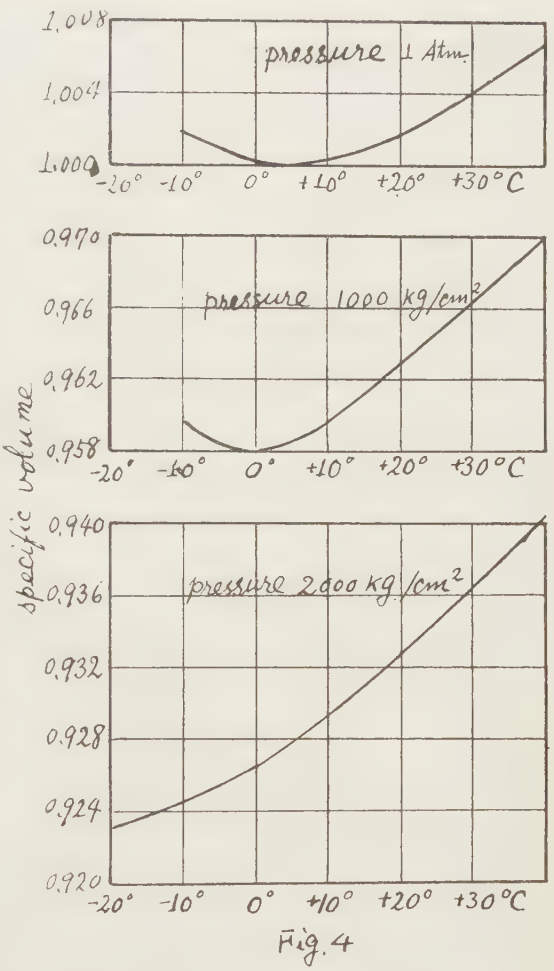


Fig. 4

Hauser[2] examined the change of the coefficient of viscosity of water with pressure at various temperatures. He measured the coefficients of viscosity of water η_{400} and η_0 at the pressures of 400 atms. and 1 atm. respectively at various temperatures. The relative percentage difference $\{(\eta_{400}-\eta_0)100\}/\eta_0$ for various temperatures are plotted in Fig. 5. As is seen by the curve in Fig. 5, the relative percentage difference becomes negative when the temperature is decreased from about 30°C. This fact seems to conform our view. When the pressure is increased at a certain temperature below 30°C, where the ice content in water is noticeable, some ice crystals melt and disintegrate into much smaller water mole-

1 Bridgman: ZS. f. anorg. Ch., **77**, 384–385 (1912)
2 Hauser: Ann. d. Phys., **5**, 597 (1901)

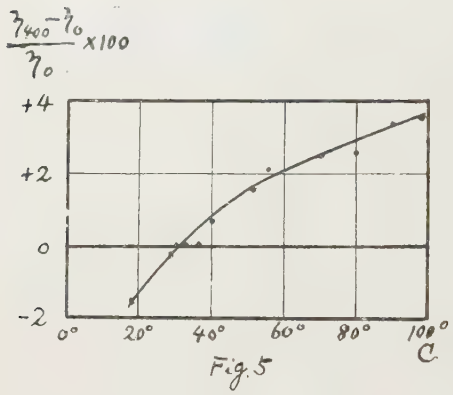


Fig. 5

cules constituting ideal liquid water; and as the result of this displacement of the ice crystals by much smaller water molecules the coefficient of viscosity of the water becomes smaller with increasing pressure at a temperature below 30°C.

Recently the writer and K. Koyanagi[1] found that when a metal was subjected for a long time to a temperature just below its melting point, minute grooves were formed at its crystal boundaries. They attributed the cause of the formation of such grooves to the fact that the metal crystal is easier to melt at its boundaries and also at its contact surface with its melt than in the interior of the crystal. This seems to be due to the irregularity of the intermolecular force at the crystal boundaries and at the contact surface of the crystal with its melt. In the case of a tiny ice crystal having the size of the order of its unit cell, its contact surface with ideal liquid water is very large when compared with its volume; and its tendency to melt is strong even in the temperature range belonging to the supercooled state below 0°C. Moreover there is a macroscopic force that enables the ice crystal to melt at a temperature above 0°C and to grow at a temperature below 0°C. Influenced by such a macroscopic force the melting tendency of the tiny ice crystal is strengthened above 0°C and is weakened below 0°C. Thus it is thought that the existence of the individual tiny ice crystals is not permanent but momentary. The growth of the ice crystal to a considerable size below 0°C is of course governed by the macroscopic force, but a stable nucleus is needed to start it. In the supercooled state, we must of course admit the existence of the tiny ice crystals having the size of the order of their unit cell; but as the supercooled state persists under suitable conditions, such tiny ice crystals are so unstable that they cannot become the nuclei of the crystal growth except at the extreme low temperature attainable in supercooled water. Thus if the freezing of water is to occur, the existence of nuclei of larger crystal fragments, whose melting tendency is less than that of

1 U. Yoshida and K. Koyanagi: These Memoirs, **18**, 9 (1935)

the smallest crystal fragments mentioned above, or the proper condition for the formation of such nuclei is necessary. If, in a supercooled state, some minute crystals of the proper size are in some way formed in water, their weak melting tendency will be overcome by the stronger macroscopic force which causes them to grow, and the whole liquid water will freeze immediately. It may be supposed, considered microscopically, that some parts of an ice crystal at a temperature below zero are in the state of liquid water just as ordinary liquid water contains some tiny ice crystals in it. But actually this is inconceivable, and the ice has a definite melting point at 0°C. This is, of course, due to the presence of a macroscopic force tending to melt the ice at a temperature above 0°C and to freeze the water at a temperature below 0°C. But, according to the writer's view, the other reason for it is that the melting of a crystal usually takes place only at its exterior boundary where the intermolecular force is irregular, and not in its interior where the intermolecular force is regular, even at a temperature which is higher than or equal to its melting point. Thus in the case of ordinary ice at a temperature below 0°C, if melting occurs spontaneously at some parts of it, it will be confined to a very thin layer at the exterior boundary of the ice. The amount of the liquid water thus formed will be negligible compared with the total mass of the ice, and the whole block of the ice crystal retains its solid state almost perfectly just up to its melting point.

If the consideration above stated that ordinary liquid water contains a small quantity of tiny ice crystals having the size of the order of their unit cell is correct, this is the universal nature of all liquids; and the liquid of any substance must be regarded as containing similarly some quantity of crystals having the size of the order of their unit cell. The increase in volume by the freezing of water into ice crystals is a peculiar characteristic of water, a peculiarity which causes the occurrence of the minimum value of a specific volume of water at 4°C. However, with most substances except water, the volume contracts by solidification from the liquid to the crystalline state; and the specific volume always increases with the rise of the temperature.

On the Fibrous Structure of Paraffin

By **Hirosi Riusan**

(Received on Nov. 15, 1936)

Abstract

Studies of the fibrous structure of paraffin which has been drawn into the shape of a wire, and of its relation to temperature are herein reported.

Long chain molecules of paraffin arrange themselves perpendicularly[1] to the surface of the thin films which were prepared by melting and pouring paraffin on a flat surface, or by vaporising the solvent from its solution on a flat surface, or on the thin film of paraffin made on a curved surface of mercury. On the other hand compressed or rolled paraffin sheets have a fibrous structure, the axis of which is perpendicular to the surface of compression and rolling[2], with chain molecules lying parallel to the fibre axis. The following results were obtained when paraffin was drawn into wire form.

Paraffin was maintained at a temperature a few degrees below its melting point, and was drawn by hand into the form of a wire in the soft state. These paraffin wires were illuminated by an X-ray beam normally to the wire axes; one of the diffraction photographs obtained is reproduced in Fig. 1, Plate I. When the photograph was taken by projecting the X-ray beam parallel to the wire axis, the diffraction pattern was of complete rings, showing that the fibre axis coincides with the axis of the wire. Fig. 2, Plate I was obtained when paraffin in the soft state described above was pushed through a die of 1 mm diameter. This photograph is almost the same as that of Fig. 1, Plate I. Paraffin used in the present experiment was of Merck's manufacture and has a melting point of 42°–44°C. Fig. 3, Plate I was obtained when paraffin, softened by soaking in ether, was pushed through a die into wire form. A similar fibre photograph was obtained when benzene was used in place of ether. The fibre photographs thus obtained were almost entirely the same as reproduced in Figs. 1 and 2 in Plate I. As can be seen on the photographs, the reflection due to the atomic

1. e. g. G. L. Clark: Applied X-rays.
2. S. Tanaka and A. Tsuji: These Memoirs, **13**, 369 (1930).

planes of the long spacings appears on the equator, the long chain molecules of paraffin do not lie parallel to the wire axis but lie in the plane perpendicular to it, and the direction of the fibre axis is approximately normal to the atomic plane which produces the most intense reflection. Fig. 4, Plate I was also obtained when paraffin softened by soaking in ether was made into wire form by the use of a die. This photograph is different from Fig. 3, Plate I. In this case the paraffin was in a very viscous state, and though the long chain molecules of paraffin lie nearly in the plane perpendicular to the wire axis, the normal to the plane which produces the most intense reflection inclines to the fibre axis at an angle of about 30°. The chief spacings were calculated by Bragg's equation and are tabulated in Table I. As is seen in the table the long spacing seems to vary more or less depending upon the methods of preparing the specimens.

Table I

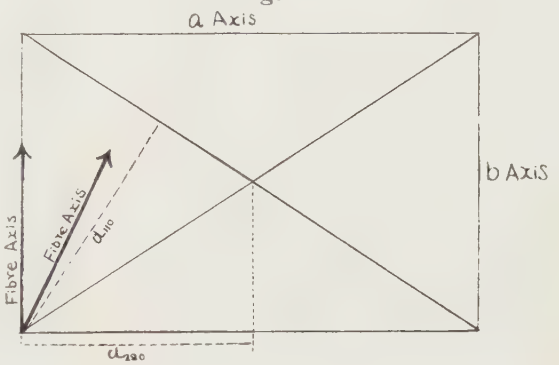
Method of preparing the specimen		Main spacing d_1	Side spacing d_2	d_3	d_4	d_5	d_6	d_7	d_8
Elongated by hand while being softened by heat		31.9	4.17	3.77	5.47	2.57	2.29	2.18	
Pushed out through a die in the softened state	after heat	27.1	4.14	3.75					
	after soaking in ether (1st form)	28.6	4.16	3.77					9.53
	after soaking in ether (2nd form)	31.6	4.17	3.77					
Index		(001)	(110)	(200)					

According to Müller[1] and Hengstenberg[2] the crystals of paraffin, $C_{29}H_{60}$ and $C_{35}H_{72}$, are orthorhombic. Plane (110) gives the strongest reflection and plane (200) next to the strongest. The spacings of these planes are 4.14 Å, 3.72 Å respectively (by Hengstenberg). These values accord with the d_2 (the strongest) and d_3 (the next strongest) in Table I. Paraffin which consists of a mixture of the higher members of the hydrocarbons with different number of carbon atoms, is supposed to build a mixed crystal whose structure is analogous to the above single pure compounds. The main reflections on the photographs then seem to be explained by taking approximately the [110] and the [010] axes

1 A. Müller: Proc. Roy. Soc. **120** A, 437 (1928).
2 J. Hengstenberg: Zeit. Krist. **67**, 583 (1928).

respectively in the above two cases. The arrows in Fig. 1 show such directions of the fibre axes.

Fig. 1



Cross Section of Unit Cell

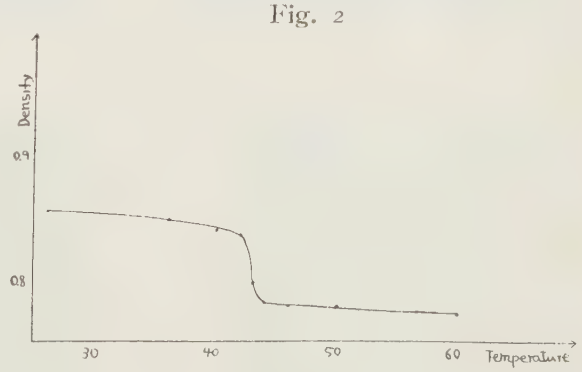
Next the effect of temperature upon the fibre structure of paraffin was studied. The specimen shown in Fig. 4, Plate I was heated at 40°C for an hour. After it was cooled to the room temperature the X-ray photograph was again taken, and no difference was found before and after the heat treatment. The same experiments were repeated by heating the specimens at 42°, 44°, and 46°C. The photographs reproduced in Fig. 5, Plate I were obtained by heating at 46°C, which is a little higher than the melting point of the paraffin used. It is clear from these photographs that the main aspect of the fibre structure is not altered by such heat treatment, though a slight tendency of the interference maxima to become diffused is perceived. This point is very interesting, and seems to point out clearly the existence of long chain molecules in paraffin.

Next the net densities of paraffin near the melting point were measured. The method used is essentially the vacuum method[1] devised by U. Yoshida and B. Takei, but somewhat

Table II

Temperature	Net density
26°C	0.856
36	0.850
40	0.841
42	0.838
43	0.799
44	0.781
46	0.780
50	0.779
60	0.773

Fig. 2



1. U. Yoshida and B. Takei: These Memoirs, **15**, 1 (1932).

adapted to the needs of this case; and a picknometer was used in the liquid state. The results observed are given in Table II and Fig. 2. As seen from Fig. 2 the density of the paraffin used decreases rapidly between 42°C and 44°C, and remains nearly the same above 44°C. This indicates that the paraffin was certainly in a liquid state at 46°C, and that the fibre structure of paraffin was still retained in the liquid state at a temperature a little above its melting point.

In conclusion, the writer wishes to express his sincere thanks to Professor U. Yoshida for his kind guidance, and also to Dr. K. Tanaka for the facilities afforded him during the present experiment.

Plate I

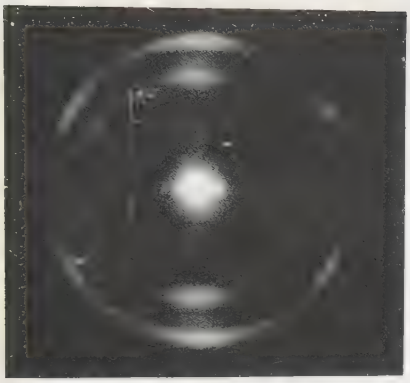


Fig. 1 (Anticathode Fe, Distance 3.95 cm)

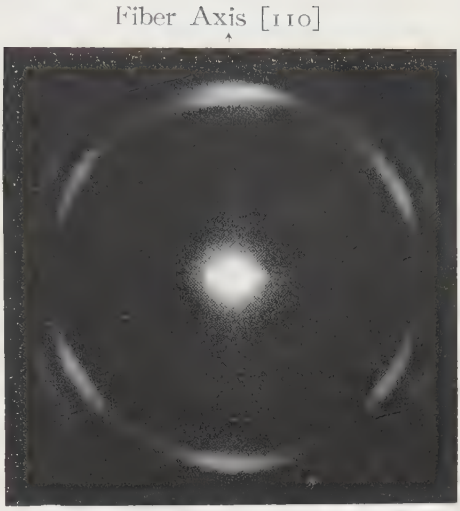


Fig. 2 (Anticathode Cr, Distance 3.95 cm)

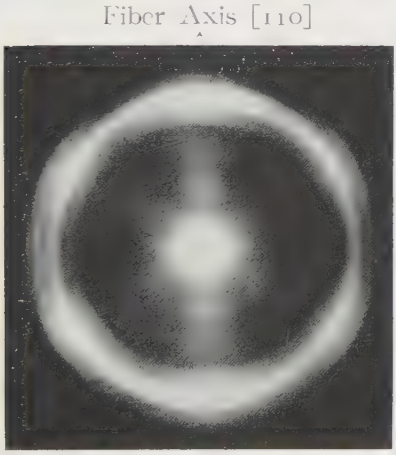


Fig. 3 (Anticathode Fe, Distance 3.95 cm)

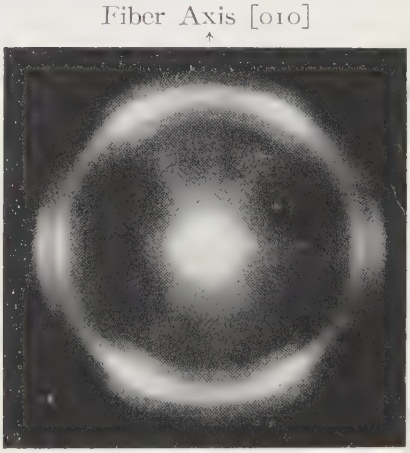
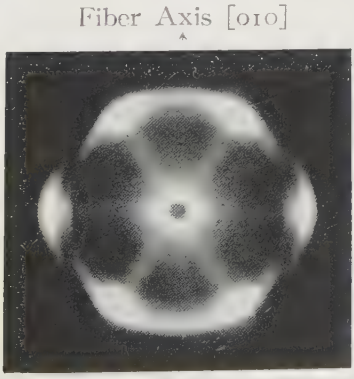
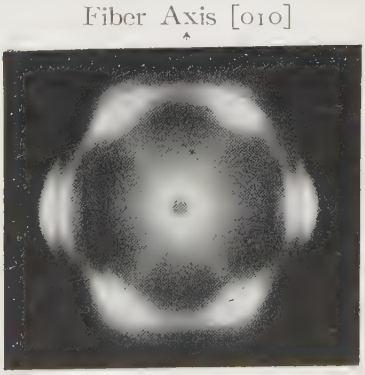


Fig. 4 (Anticathode Fe, Distance 3.95 cm)



(a) Before heat treatment



(b) After heat treatment

Fig. 5 (Anticathode Fe, Distance 2.95 cm)

Experimental Investigation on the Sound Emitted by Japanese Hanging-Bells

By Ichiro Aoki

(Received Nov. 16, 1936)

Abstract

The writer recorded the sounds of five bells which belong to temples in and near Kyoto by using a Low-Hilger audiometer. These records of sounds were analysed into partial sine-curves by the method of periodogram analysis. Taking the results of former investigations into account, the writer arrived at the following conclusions:

1. The frequencies of the fundamental tones are nearly in inverse proportion to the linear dimensions of the bells. But as the bell becomes smaller, the frequency of its fundamental tones becomes smaller than the value obtained by this relation.

2. The frequencies of the partial tones do not change when we change the position of the striking-point along the same meridian line passing through the "Tsukiza" (striking-seat). But their values change when we strike different points along the same horizontal line passing through the "Tsukiza".

3. There exists one partial tone between the fundamental tone and what was formerly considered the first overtone. This overtone seems to appear when the size of the bell becomes comparatively large.

4. The frequency of the fundamental tone and its overtones are in approximately the following ratios:

$$4^2 : 5^2 : 6^2 : 7^2 : 8^2 : \ldots\ldots$$

1. The Method of Investigation

The sound of some Japanese hanging-bells was recorded on rotating films by a Low-Hilger audiometer, and analysing curves thus obtained by periodogram analysis,[1] the frequency and amplitudes of the partial tones, which constitute the sound of bells, were determined.

The bells newly investigated are those which belong to the following temples:

A. Byôdôin-temple in Uji-chô.
B. Hônenin-temple in Kyoto.
C. Zenrinji-temple in Kyoto.
D. Kakôji-temple in Kyoto.
E. Daiunji-temple in Iwakura-mura.

We will hereafter call these five bells A, B, C, D and E respectively.

1. Whittaker and Robinson, "The Calculus of Observation." pp. 343-360.
Carse and Shearer, "Fourier's Analysis and Periodogram Analysis." pp. 29-50.

The photographs and the dimensions expressed in cm. of these five bells are shown in Pls. I, II, and III. The results already obtained by K. Yamashita, J. Obata and the present writer are added and treated together. Thus in all, nine bells are treated in this paper.

Size	Big Bells						Small Bells		
Designation of Bell	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Name of Temple	Byôdôin	Hônenin	Zenrinji	Kakôji	Daiunji	Myôshinji			
Investigator	Aoki	Aoki	Aoki	Aoki	Aoki	Aoki[1]	Obata[2]	Aoki[3]	Yamashita[4] and Aoki

2. The Records of the Sounds

The sound-records of the five bells *A*, *B*, *C*, *D*, *E* are shown in Pls. IV, V, VI, VII, and VIII respectively. Fig. 1's in these five plates are records of the sounds taken immediately after the bell was struck, and Fig. 2's and Fig. 3's are the records taken some minutes after the striking. The last figures in these plates are the records of the sounds taken when the bell was struck with a very small wooden hand-hammer (usually used for small bells) thickly wrapped in cloth.

The bell *E* has a somewhat different form, its shape being nearly cylindrical, and the ratio of the height to its diameter being much larger than those of the other bells. So the writer recorded the sounds of this bell by varying the point of striking in order to investigate the relation between the frequency of the partial tones and the position of the striking. These records are shown in Pl. IX. Figs. 1, 2 and 3 in this plate are the records taken respectively when the bell was struck at points 10 cms. above, 20 cms. above and 10 cms. below the striking-seat along the same meridian line. Figs. 4, 5, 6 and 7 are the records made when the bell was struck at points $\frac{1}{4}90^\circ$, $\frac{1}{2}90^\circ$, $\frac{3}{4}90^\circ$ and 90° apart from the striking seat along the same horizontal line. Each of these figures in Pl. IX are the records taken 2–4 secs. after the bell was struck.

The marked portions in these six plates were measured accurately and were analysed into partial sine-curves. The results of this analysis

1. I. Aoki, These Memoirs, XV, 311–313 (1932).
2. J. Obata and T. Tesima, Japanese Journal of Physics, IX, 49–73 (1934).
3. I. Aoki, These Memoirs, XIV, 213–218 (1931).
4. K. Yamashita and I. Aoki, loc. cit. XV, 323–326 (1932).

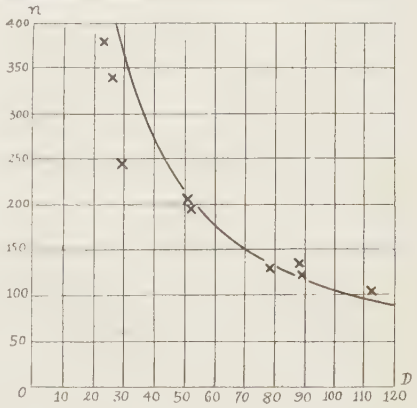
are shown in the diagrams appended to this paper. These diagrams show the frequencies and the amplitudes of the partial tones of the five bells.

The frequencies in these diagrams having noughts as their last figures were not so accurately obtained as the others, owing to the method of the periodogram analysis.

3. On the Fundamental Tones

Between the frequencies of the fundamental tones, n, and the diameters of the bells at their mouths, D, we compiled the following table and graph.

	A	*B*	*C*	*F*	*D*	*E*	*G*	*H*	*I*
D	112.0	88.5	88.0	79.5	51.0	50.5	28.5	25.6	22.7
n	108	125	135	129	197	206	247	340	380
nD	12096	11062	11880	10255	10047	10403	7039	8704	8626



It is evident theoretically that if the bells have similar forms (including thickness) and are made of the same material, n will be in inverse-proportion to D, i. e. the product nD will be constant and the graph will become a rectangular hyperbola[1]. But in our case, if we draw a rectangular hyperbola nearly through the mean positions of n, we find that the points showing the frequencies of the larger bells lie nearly on the curve, while those of the smaller bells deviate much from the curve on its lower side. It seems this is due to the fact that the forms of the bells are not quite similar, especially in the way the thickness of each bell varies.

In this graph, we see that the position of n of the bell *G* most deviate from the mean curve. Taking the ratios of the thickness d at the lowest ends of the bells to the diameter D at their mouths, we have the following table, which shows that d is nearly equal to 10%

1. K. Yamashita, These Memoirs, XV, 315-322 (1932).
K. Yamashita and I. Aoki, loc. cit. XV, 323-326 (1932).

of D and that there must be some remarkable deviation in the bells F and G from the others. In fact, the bell F has no specially thick portion at its bottom end usually called "Komanotsume", and the bell G has a frequency n much lower than that given by the hyperbolic curve. From this table we see that the values of $(d/D) \times 10$ for different

	A	B	C	F	D	E	G	H	I
D	112.0	88.5	88.0	79.5	51.0	50.5	28.5	25.6	22.7
d	11.5	10.0	11.0	5.0	6.0	4.8	1.6	2.6	2.3
$\frac{d}{D} \times 10$	1.03	1.13	1.25	0.63	1.18	0.95	0.56	1.02	1.01

bells are nearly equal to unity, i. e. the thickness of the bells is nearly equal to 10% of the diameter, exceptions being in the case of the bells F and G. But the bell F does not have a so-called "Komanotsume" (the lowest thick part), and the thickness along the meridian line does not decrease so rapidly as do the other bells as we pass upward from the lowest end. For this reason, it seems, the bell G will show the greatest deviation, but the bell F will not deviate so much. The deviation of smaller bells G, H, and I will be explained by the fact that in their cases the thickness along the meridian line decreases more rapidly as we pass upwards from the lowest end, and the upper parts of the "Komanotsume" are much thinner than those of the larger bells.

4. The Relation between the Position of the Striking-point and the Frequencies of the Partial Tones

The bell E is, as we have already stated, nearly cylindrical and the ratio of its height to its diameter is much larger than those for the other bells. Therefore, with it the writer tried to investigate the relation between the position of the striking-point and the frequencies of the partial tones by shifting the striking-point along the meridian line and the horizontal line passing through the striking-seat. The result is as follows:

	Position of Striking	Fundamental Tone	First Overtone
Meridian Line	Striking-seat	206	450
	10 cms. above	206	450
	20 cms. above	206	450
	10 cms. below	206	450

(Continued from the preceding page)

	Position of Striking	Fundemental Tone	First Overtone
Horizontal Line	$\frac{1}{4}$ 90°	206	450
	$\frac{1}{2}$ 90°	206	**445**
	$\frac{3}{4}$ 90°	**207**	450
	90°	206	**452**

From these findings we see that when the position of the striking-point is displaced along the same meridian line the frequency of the partial tone does not change, while when the point is displaced along the horizontal line, the frequency generally changes a little, as the writer has previously reported in the case of the small bell.[1]

5. On the Frequencies of the Partial Tones

Up to this time in regard to the ratios of the frequencies of the partial tones of the bell, the following relations were proposed:

$2^2 : 3^2 : 4^2 : 5^2 : 6^2 : \ldots\ldots$[2]

(proposed by K. Yamashita and the present writer)

$2 : 5 : 7 : 9 : 11 : \ldots\ldots$[3]

(proposed by J. Obata and T. Tesima)

But in the former relation, the fourth partial tone whose frequency may come between 4^2 and 5^2 is excluded.

Now all the frequencies of the partial tones of the nine bells we have found are shown in the following table.

Designation of Bell	Frequencies of the Partial Tones										
A	108	200	276	333	450	830	1600				
B	125	251	330	530	660	900	1250	1600	2900		
C	135	220	300	390	450	690	1000	1300	1500	1900	2400
D	197	430	720	930	1200	1500					
E	206	450	740	950	1250	1600					
F	129	335	550	700	900	1250					
G	247	624	850	1100	1380	1640	2050				
H	340	800	1100	1400	1800						
I	380	900	1250	1600	2450	3600					

1. I. Aoki, These Memoirs, XIV, 213-218 (1931).
2. K. Yamashita and I. Aoki, These Memoirs, XIV, 213-218 (1931).
3. J. Obata and T. Tesima, Japanese Journal of Physics, IX, 49-73 (1934).

If we take the frequencies of the fundamental tones as 10, the ratios of the frequencies of the higher partial tones are shown in the following table.

Designation of Bell	Ratios of Frequencies										
A	10	18	26	31	42	77	150				
B	10	20	26	42	53	72	100	128	232		
C	10	16	22	29	33	51	74	96	111	147	178
D	10	22	36	48	61	79					
E	10	22	36	46	61	78					
F	10	26	43	52	70	97					
G	10	25	34	44	56	66	83				
H	10	23	32	41	53						
I	10	24	33	42	64	95					

Rearranging these numbers under the supposition that some of the partial tones might be absent, we get:

Table A.

Order of Partial Tones / Designation of Bell	1	2	3	4	5	6	7
F	10		26		43	52	70
B	10	20	26		42	53	72
A	10	18	26	31	42		
G	10		25	34	44	56	66
I	10		24	33	42		64
H	10		23	32	41	53 →	?
C	10	16	22	29	33	51	
D	10		22		36	48	61
E	10		22		36	46	61
Mean	10	18	24	32	40	51	64

If we put the frequencies of the third partials equal to 10, the above table becomes:

Table B.

Order of Partial Tones / Designation of Bell	1	2	3	4	5	6	7
F	3.7		10		16	21	27
B	3.8	7.9	10		16	20	27
A	3.9	7.2	10	12	16		
G	4.0		10	14	18	22	26

(Continued from the preceding page)

Designation of Bell \ Order of Partial Tones	1	2	3	4	5	6	7
I	4.2		10	14	18		27
H	4.2		10	14	18	22	
C	4.3	7.3	10	13	15	23 →	?
D	4.6		10		17	22	28
E	4.8		10		16	21	28
Mean	4.2	7.5	10	13	17	21	27

Similarly by putting the frequencies of the fifth partials equal to 10, we get:

Table C.

Designation of Bell \ Order of Partial Tones	1	2	3	4	5	6	7
F	2.3		6.1		10	13	16
B	2.4	4.7	6.3		10	12	17
A	2.2	4.4	6.1	7.2	10		
G	2.2		5.7	7.6	10	13	15
I	2.4		5.0	7.8	10		15
H	2.4		5.7	7.9	10	13	
C	3.0	4.9	6.1	8.7	10	15 →	?
D	2.7		6.0		10	13	17
E	2.8		6.1		10	13	17
Mean	2.5	4.7	5.9	7.8	10	13	16

By these table, we see that the numbers in each column are nearly equal, a fact which may be regarded as proof that they are of the partial tones of the same order. Rendering these numbers into frequencies, we get:

Designation of Bell \ Order of Partial Tones	1	2	3	4	5	6	7
A	108	200	276	333	450		
C	135	220	300	390	450	690 →	?
B	125	251	330		530	660	900
F	129		335		550	700	900
D	197		430		720	930	1200
E	206		450		740	950	1250
G	247		624	850	1100	1380	1640
H	340		800	1100	1400	1800	
I	380		900	1250	1600		2450

Taking the mean values of each columns in table A, B, C, we get:

Table \ Order of Partial Tones	1	2	3	4	5	6	7
A	10	18	24	32	40	51	64
B	4.2	7.5	10	13	17	21	27
C	2.5	4.7	5.9	7.8	10	13	16

Keeping the same ratios and changing the numbers in the first column to 10, we get:

Table \ Order of Partial Tones	1	2	3	4	5	6	7
A	10	18	24	32	40	51	64
B	10	18	24	31	40	50	64
C	10	19	24	31	40	52	64
Mean	10	18	24	31	40	51	64

If we assume the ratios

$$4^2 : 5^2 : 6^2 : 7^2 : \ldots\ldots\ldots$$

modifying Dr. Yamashita's relation a little, and the ratios

$$2 : 3 : 5 : 7 : \ldots\ldots\ldots$$

adding one partial between the first and the second in Dr. Obata's relation, and dividing the above mean values by the corresponding numbers in these ratios, we get:

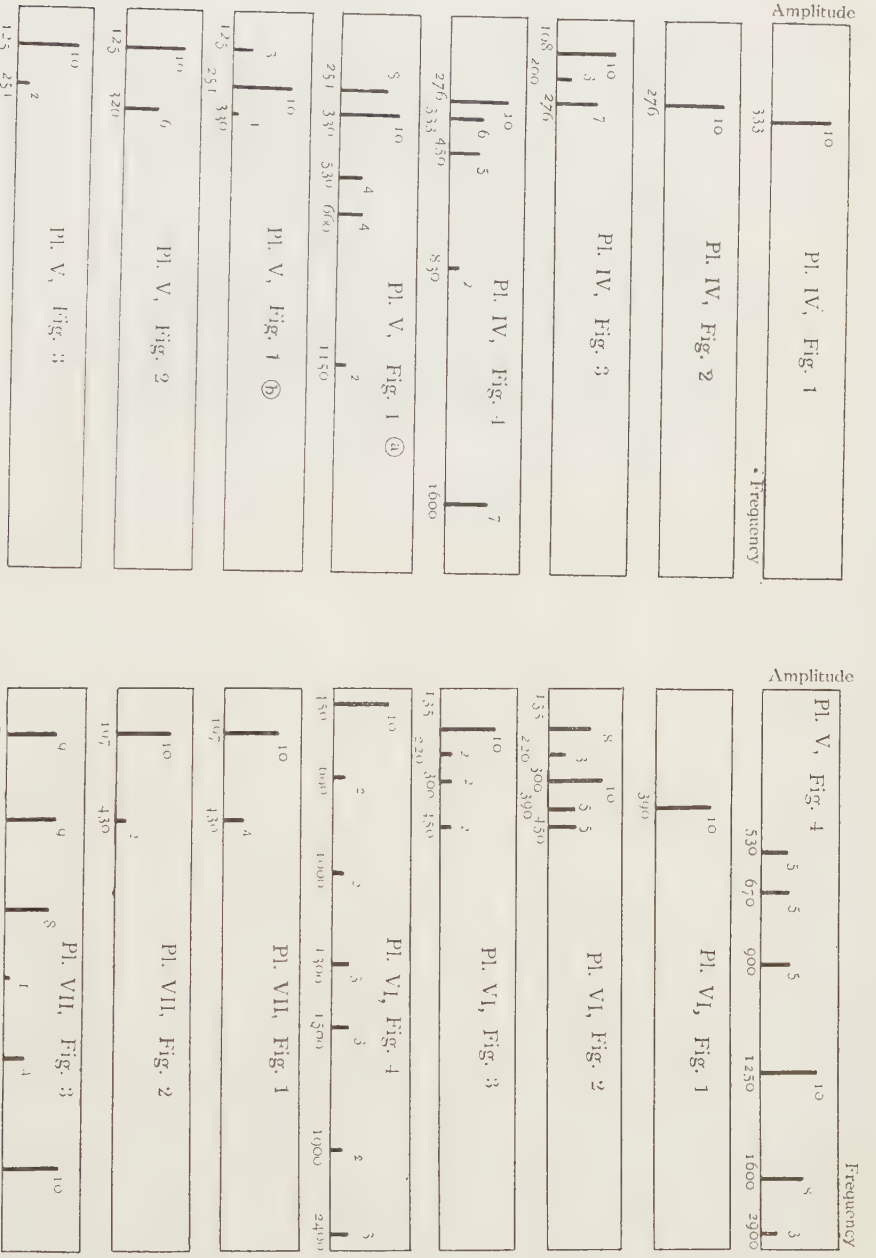
$10 \div 4^2 = 0.63$ $\quad$ $10 \div 2 = 5.0$
$18 \div 5^2 = 0.72$ $\quad$ $18 \div 3 = 6.0$
$24 \div 6^2 = 0.67$ $\quad$ $24 \div 5 = 4.8$
$31 \div 7^2 = 0.63$ $\quad$ $31 \div 7 = 4.4$
$40 \div 8^2 = 0.63$ $\quad$ $40 \div 9 = 4.4$
$51 \div 9^2 = 0.63$ $\quad$ $51 \div 11 = 4.6$
$64 \div 10^2 = 0.64$ $\quad$ $64 \div 13 = 4.9$

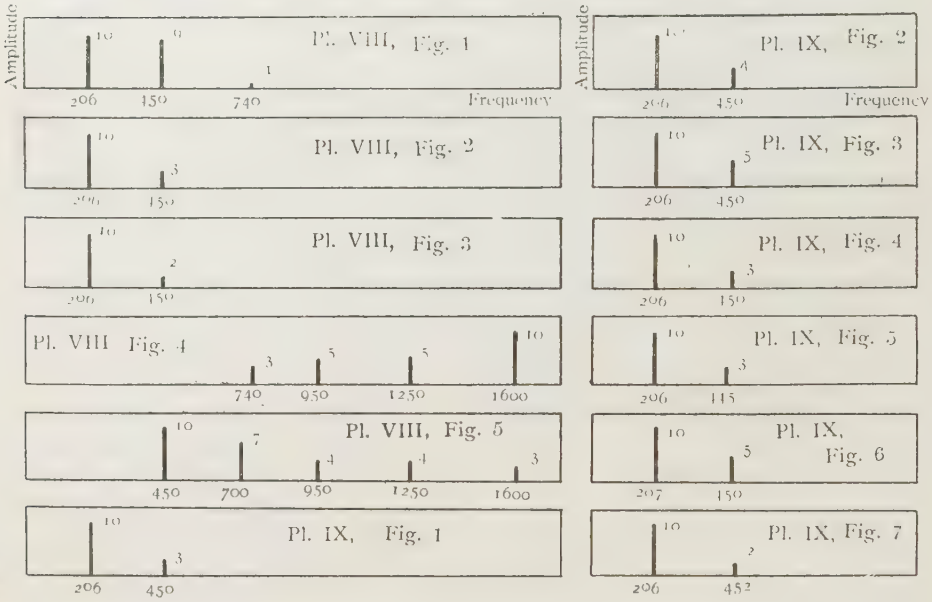
These quotients are nearly constant in both cases, but the agreement is better in the former ratios. Thus the frequencies of the partial tones of the bell may be represented by the ratios

$$4^2 : 5^2 : 6^2 : 7^2 : \ldots\ldots\ldots$$

and the second partial tones only appear in larger bells *A*, *B*, and *C*.

In conclusion the writer wishes to express his gratitude to Prof. K. Tamaki for his kind guidance, to Dr. K. Yamashita for his advice and to the Hattori Hôkô-kwai for the grant with which the present investigation has been carried out. The writer also cordially thanks the authorities of the temples who kindly permitted him to use the bells for the present study.


Amplitude
Frequency
Pl. IV, Fig. 1
Pl. IV, Fig. 2
Pl. IV, Fig. 3
Pl. IV, Fig. 4
Pl. V, Fig. 1 (a)
Pl. V, Fig. 1 (b)
Pl. V, Fig. 2
Pl. V, Fig. 3
Pl. V, Fig. 4
Pl. VI, Fig. 1
Pl. VI, Fig. 2
Pl. VI, Fig. 3
Pl. VI, Fig. 4
Pl. VII, Fig. 1
Pl. VII, Fig. 2
Pl. VII, Fig. 3


Amplitude
Pl. VIII, Fig. 1
10
6
1
206
450
740
Frequency
Pl. VIII, Fig. 2
10
3
206
450
Pl. VIII, Fig. 3
10
2
206
450
Pl. VIII Fig. 4
3
5
5
10
740
950
1250
1600
Pl. VIII, Fig. 5
10
7
4
4
3
450
700
950
1250
1600
Pl. IX, Fig. 1
10
3
206
450
Amplitude
Pl. IX, Fig. 2
4
206
450
Frequency
Pl. IX, Fig. 3
10
5
206
450
Pl. IX, Fig. 4
10
3
206
150
Pl. IX, Fig. 5
10
3
206
145
Pl. IX, Fig. 6
10
5
207
150
Pl. IX, Fig. 7
10
2
206
452

Ichiro Aoki

Plate I

(Bell *A*)

Fig. 1

Fig. 1′

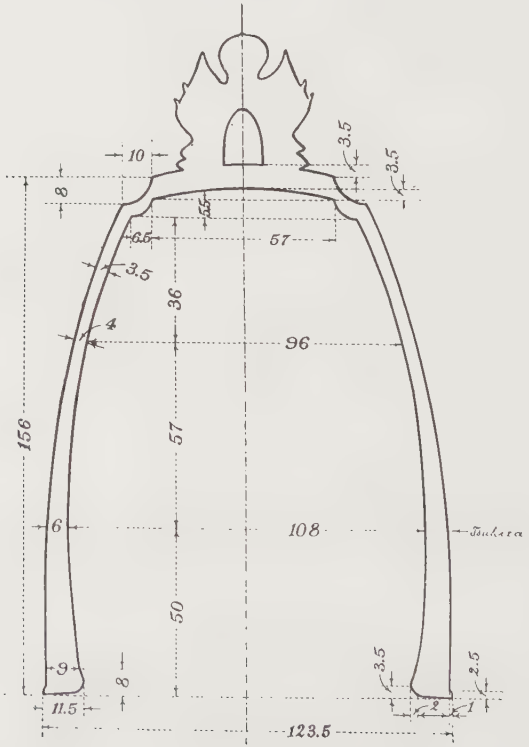


Plate II

(Bell *B*)

Fig. 1

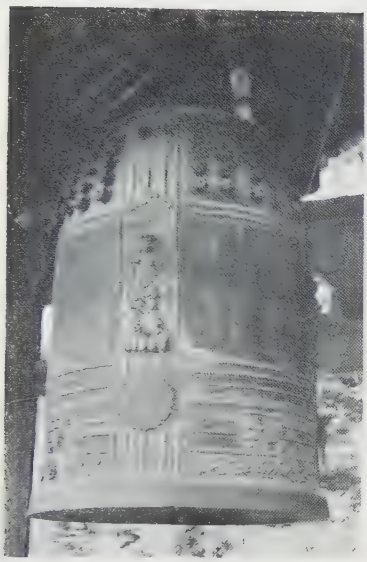


Fig. 1′

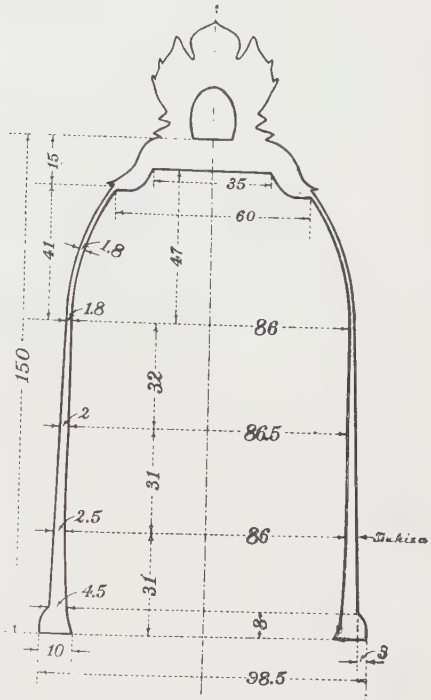


Fig. 2′ (Bell *C*)

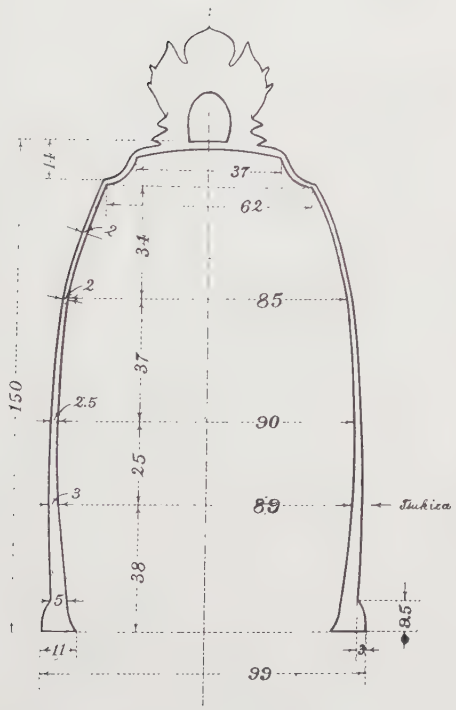


Fig. 2

Plate III

Fig. 1 (Bell *D*)

Fig. 1′

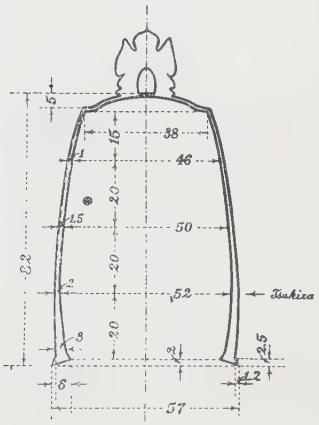


Fig. 2 (Bell *E*)

Fig. 2′

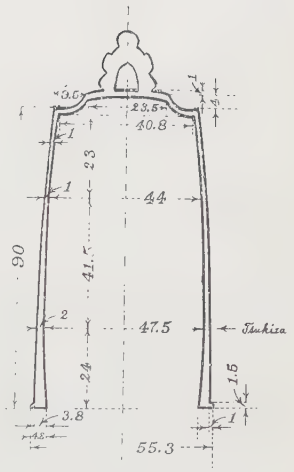


Plate IV (Bell *A*)

Fig. 1.

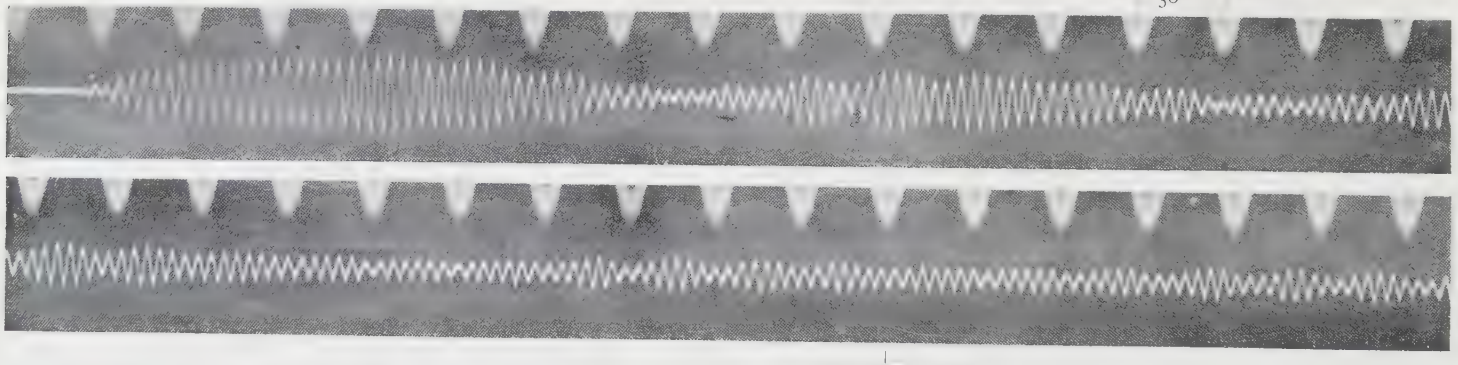


Fig. 2. 1 sec. after it is struck.

Fig. 3. 1 sec.

Fig. 4.

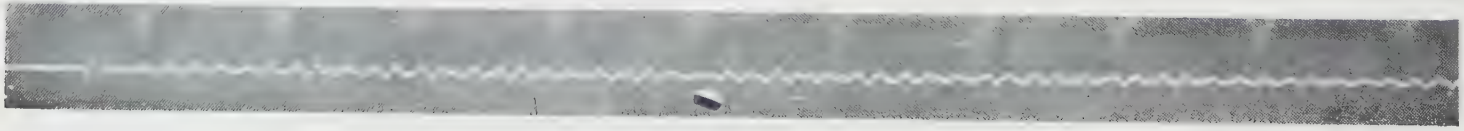


Plate V (Bell *B*)

Fig. 1.

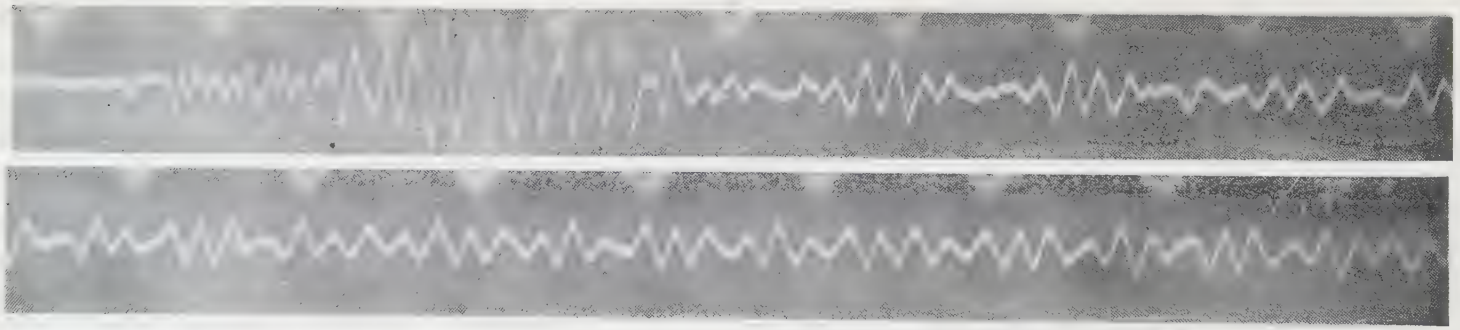


Fig. 2. 2 secs.

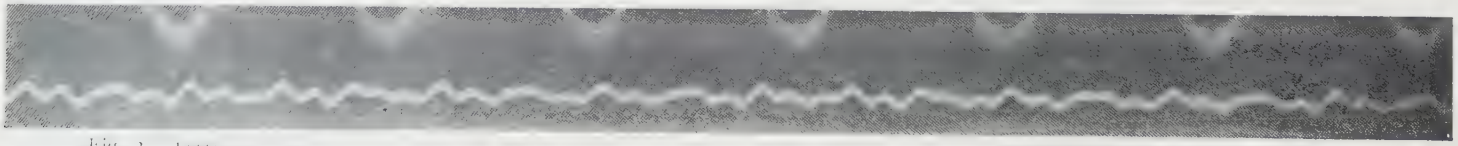


Fig. 3. 1 sec.

Fig. 4.

Plate VI (Bell *C*)

Fig. 1.

Fig. 2. 1 sec.

Fig. 3. 4 secs.

Fig. 4.

Plate VII (Bell *D*)

Fig. 1.

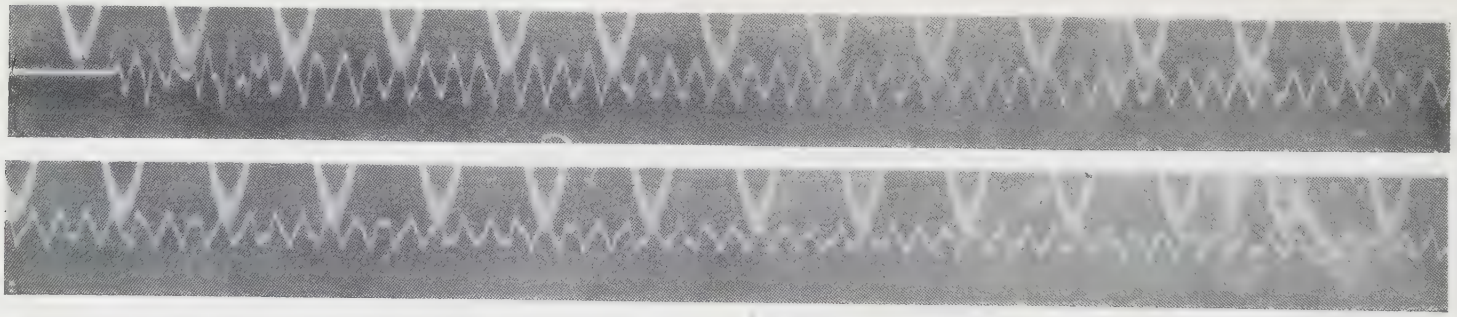


Fig. 2. 4 secs.

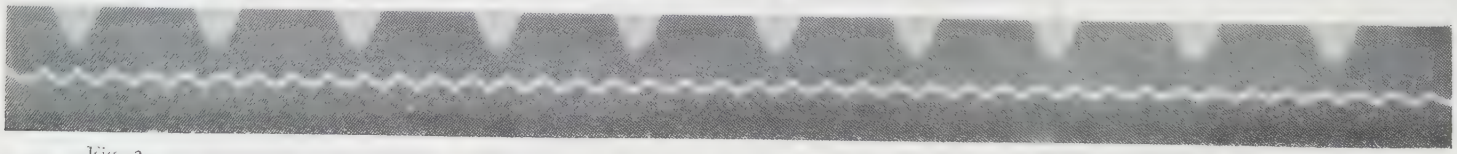


Fig. 3.

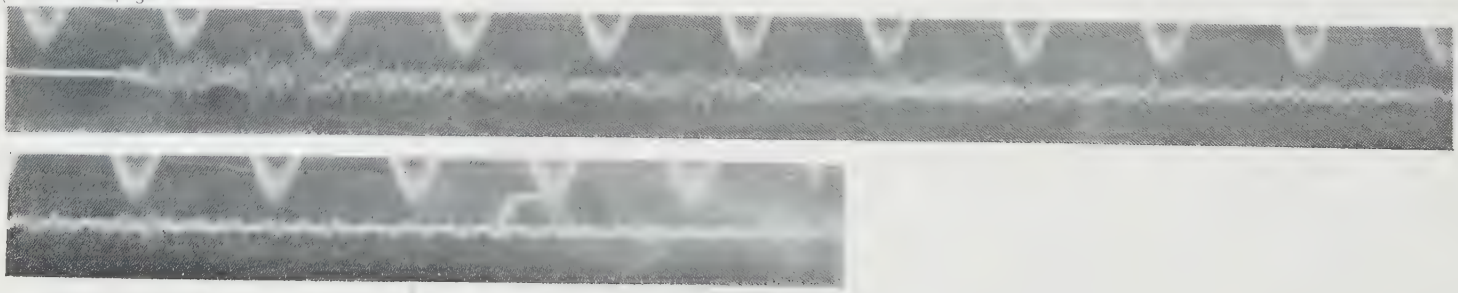


Plate VIII (Bell *E*)

Fig. 1.

Fig. 2. 2 secs.

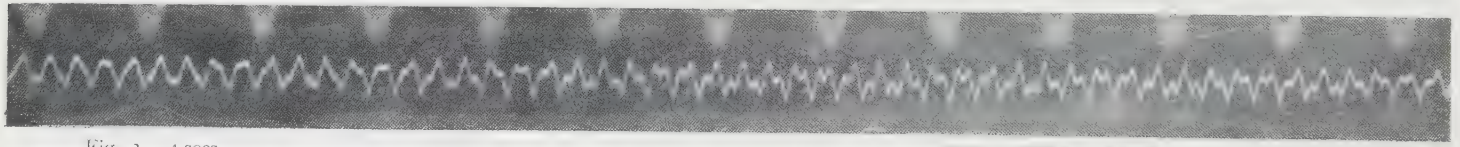


Fig. 3. 4 secs.

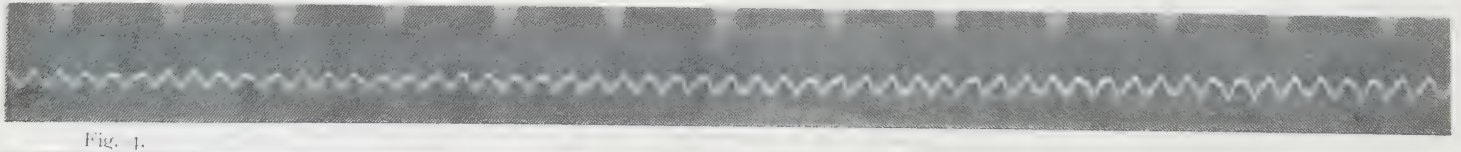


Fig. 4.

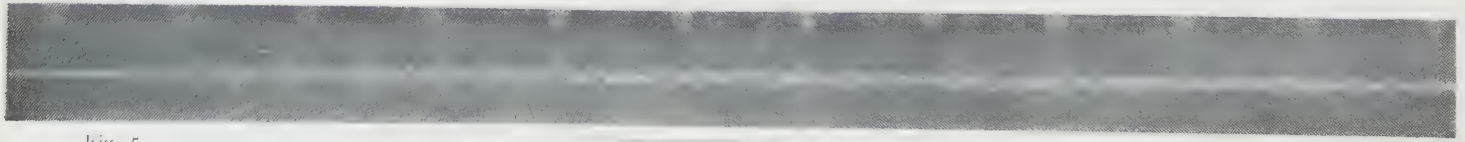


Fig. 5.

Plate IX (Bell *E*)

Fig. 1.

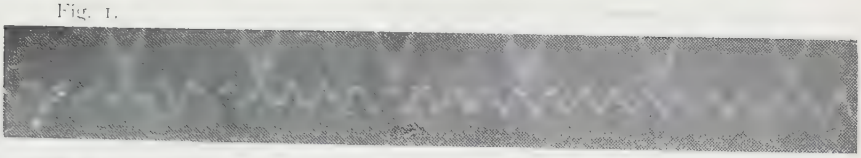


Fig. 2.

Fig. 3.

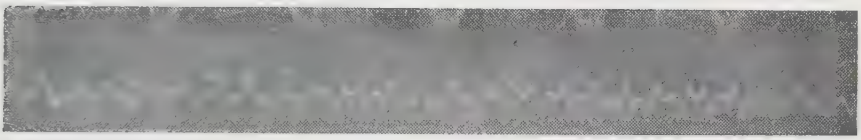


Fig. 4.

Fig. 5.

Fig. 6.

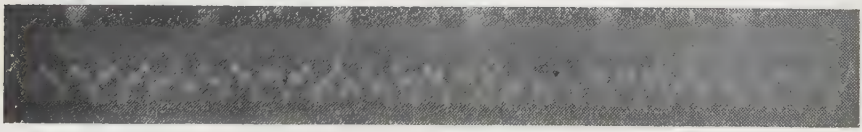


Fig. 7.

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